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## Critical values in continuum and dependent percolation

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# Critical values in continuum and dependent percolation

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June 2011

A Thesis presented for the degree of Doctor of Philosophy

Department of Mathematical Sciences

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## **Abstract**

In the first part of this thesis I consider site and bond percolation on a Random Connection Model and prove that for a wide range of connection functions the critical site probability is strictly greater than the critical bond probability and use this fact to improve previously known non-strict inequalities to strict inequalities. In the second part I consider percolation on the even phase of a Random Sequential Adsorption model and prove that the critical intensity is finite and strictly bigger than 1. Both of these main results make use of an enhancement technique.

## **Declaration**

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at the University of Bath, England, and the Department of Electrical and Computer Engineering, University of California, San Diego, USA. No part of this thesis has been submitted elsewhere for any other degree or qualification. It is all my own work unless referenced to the contrary in the text. Parts of Chapters 5-7 are adapted from joint work with Mathew D. Penrose and Massimo Franceschetti [8]. Parts of Chapter 9 are adapted from joint work with Mathew D. Penrose [20].

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# 1 Introduction

This thesis contains results in two main areas. The first area is on random geometric graphs. Given a homogeneous Poisson process a random connection model (RCM) can be formed by joining points in the Poisson process with a probability depending only on the distance between the points. If the Poisson process is of high enough intensity there will be an infinite component of the RCM on which site and bond percolation can be performed. In the first part of the thesis I show that the critical bond probability is strictly less than the critical site probability for a wide range of RCMs. Results are also given that show that multiplying the connection function by a constant less than one makes the critical intensity strictly greater. Together these results can be used to show that a ‘squashing transformation’ on the connection function makes the critical intensity strictly smaller without changing the mean degree of a vertex. All these results are improvements on non-strict inequalities that were previously known. The proofs are based on using enhancements. In earlier chapters the enhancement will be described along with a continuous version of the Margulis-Russo formula which will relate partial derivatives of percolation probabilities to pivotal points.

The second part of the thesis is concerned with random sequential adsorption (RSA). Starting with an empty square lattice of sites, each site has an independent arrival process with rate 1 on odd sites and  $\lambda$  on even sites. If a site has no occupied sites adjacent to it when its first arrival occurs then it becomes occupied and remains so, otherwise it becomes blocked and remains so. In this way every site ends up being blocked or occupied and the graph of even occupied sites and odd blocked sites can be looked at. There will be a critical arrival rate above which there will almost surely be an infinite component. In this thesis I show that this critical rate is strictly greater than 1. Again this will use an enhancement technique. It relies on a weak version of the RSW theorem that was proved by Bollobas and Riordan [3], and on the Harris-FKG inequality for this model which was proved by Penrose and Sudbury [21].

## 2 Continuum Percolation

This section introduces percolation on the Random Connection Model which is the subject of the first half of the thesis.

### 2.1 The $d$ -dimensional Poisson Process

A  $d$ -dimensional homogeneous Poisson Process of intensity  $\lambda$  is a point process  $X \in \mathbb{R}^d$  that satisfies the following properties:

- a) For mutually disjoint sets  $A_1, \dots, A_k$ , the random variables  $X(A_1), \dots, X(A_k)$  are mutually independent (where  $X(A)$  is the number of points of  $X$  in  $A$ ).
- b) For any set  $A \in \mathbb{R}^d$  we have for every  $k \geq 0$

$$P(X(A) = k) = \exp(-\lambda l(A)) \frac{(\lambda l(A))^k}{k!}$$

where  $l(A)$  is the Lebesgue measure of  $A$  in  $\mathbb{R}^d$ .

So the number of points in a set  $A$  has a Poisson distribution with mean  $\lambda l(A)$  and is independent of the process outside of  $A$ .

### 2.2 Random Connection Model

Given a point process  $X$  in  $\mathbb{R}^d$  and a connection function  $f : \mathbb{R} \rightarrow [0, 1]$ , we can form a *Random Connection Model* (RCM) as follows. For any points  $x$  and  $y$  in the process, we put an edge between them with probability  $f(|x - y|)$ , independently of everything else. In this thesis I consider Random Connection Models where  $X$  is a homogeneous Poisson Process and the connection function is non-increasing. A special case of this is Gilbert's graph where the connection function is ;

$$f(r) = I_{r \in [0, 1]}.$$

So points have an edge between them if they are within distance 1 of each other. Throughout this thesis, for any finite set of points  $A$  and a point  $x$  in  $\mathbb{R}^d$  define:

$$p(x, A) := 1 - \prod_{a \in A} [1 - f(|x - a|)]$$

to be the probability that  $x$  is joined by least one edge to  $A$ .

## 2.3 Site and Bond Percolation

Given an infinite connected graph  $G$  we can perform site percolation by independently declaring each site to be open with probability  $p$ . We let  $G(p)$  be the resulting induced subgraph containing only the open sites. If we fix a vertex of  $G$  to be the origin then we can consider  $C(p)$ , the component of  $G(p)$  which contains the origin. We let  $\theta(p)$  be the probability that  $|C(p)| = \infty$ .

**Proposition 2.1** *The percolation probability  $\theta$  is non-decreasing in  $p$ .*

**Proof:** For every vertex  $v_i$  in  $G$  we can assign a uniform random variable  $U_i \in [0, 1]$  and then form the site percolation model by declaring  $v_i$  to be open in  $G(p)$  if  $U_i < p$  and closed otherwise. In this way we get realisations of  $G(p)$  for all  $p \in [0, 1]$  with  $C(p) \subset C(q)$  if  $p \leq q$ . Therefore

$$\theta(p) \leq \theta(q)$$

if  $p \leq q$ . □

We define  $p_c^{site} := \sup\{p : \theta(p) = 0\}$ . If we have  $\theta(p) > 0$  then by Kolmogorov's 0 – 1 law we must have an infinite component of  $G(p)$  almost surely. If we have  $\theta(p) = 0$  then we must almost surely have no infinite component, as  $G$  is connected.  $p_c^{site}$  is the critical site probability. Note that while  $\theta(p)$  depends on the choice of origin,  $p_c^{site}$  is independent of the choice of origin.

Similarly we can have bond percolation where we have each bond open with probability  $p$ . We let  $\theta^*(p)$  be the probability that there is an infinite component containing the origin in this model, and we define  $p_c^{bond}$  to be the critical bond probability.

In general, exact values of  $p_c^{site}$  and  $p_c^{bond}$  are hard to find. A classic result in percolation theory is that  $p_c^{bond} = 0.5$  on the square lattice  $\mathbb{Z}^2$ . This was first proved by Kesten [12]. On the triangular lattice  $p_c^{site}$  is known to be 0.5.

Let  $X$  be a 2-dimensional homogeneous Poisson point process on  $\mathbb{R}^2$ . For  $x \in X$  define  $V_x$  to be the set of all points  $z$  in  $\mathbb{R}^2$  such that

$$|x - z| \leq |y - z| \quad \forall y \in X.$$

This is the *Voronoi cell* of  $x$ . We can then form a graph by joining all  $x, y \in X$  such that  $V_x \cap V_y \neq \emptyset$ . This is the *Delaunay Graph*. On the Delaunay Graph it has been shown that  $p_c^{site} = 0.5$  (Bollobas and Riordan [3]).

**Proposition 2.2** *On any infinite connected graph  $G$ , we have the inequality*

$$p_c^{bond}(G) \leq p_c^{site}(G).$$

**Proof.** A proof along these lines appears in Theorem 2.2.7 of [6]. We fix an origin in  $G$ . We then couple the component containing the origin in the site percolation model,  $C$ , with the component containing the origin in the bond percolation model,  $C^*$ . We order the vertices in some way  $v_1, v_2, \dots$  and order the edges in some way  $e_1, e_2, \dots$ . We let  $U_1, U_2, \dots$  be a series of uniform  $[0, 1]$  random variables. We can form the bond percolation model by having the edge  $e_i$  open if  $U_i < p$ . We then form a coupled component containing the origin in the site percolation model as follows. Set the origin to be closed. Then look for the first vertex  $v_i$  adjacent to the origin that we have not already examined and if the edge  $0 \sim v_i$  is open then set  $v_i$  to be open and if not then set  $v_i$  to be closed. If  $v_i$  is open then add it to  $C$ . In general we look for the first unexamined vertex  $v_j$  adjacent to  $C$  and then if the first edge between  $C$  and  $v_j$  is open then set  $v_j$  is open and if not then  $v_j$  is closed. We continue in this way and stop if there are no more unexamined vertices adjacent to  $C$ . Therefore the size of  $C$  has the same distribution as the size of  $C$  in a site percolation model, and if  $|C| = \infty$  then that means  $|C^*| = \infty$  in the bond percolation model. Therefore we have

$$\theta(p) \leq \theta^*(p)$$

which gives us the result.  $\square$

In general, graphs can have  $p_c^{site} = p_c^{bond}$ . For instance the infinite tree  $T_2$  has  $p_c^{site} = p_c^{bond} = 0.5$ , and indeed any infinite tree has  $p_c^{site} = p_c^{bond}$ .

Here is a table of a few infinite graphs and their site and bond percolation probabilities (where  $(3^6)$  is the triangular lattice as shown in Figure 2.3);

Graph	$p_c^{site}$	$p_c^{bond}$
$T_2$	0.5	0.5
$\mathbb{Z}^2$	0.59...	0.5
$(3^6)$	0.5	0.347...

## 2.4 Percolation on the Random Connection Model

Let  $X$  be a  $d$ -dimensional homogeneous Poisson point process of intensity  $\lambda$ . For  $x$  and  $y$  in  $X$ , let us say that  $x$  is *path-connected* to  $y$  if there is a

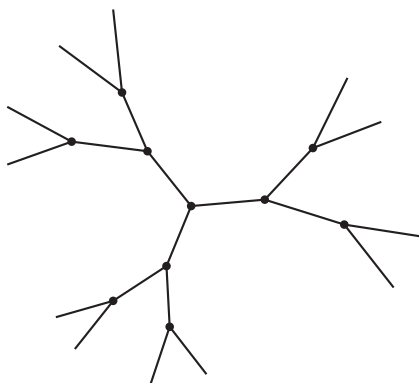


Figure 2.1: Example: Part of the tree  $T_2$

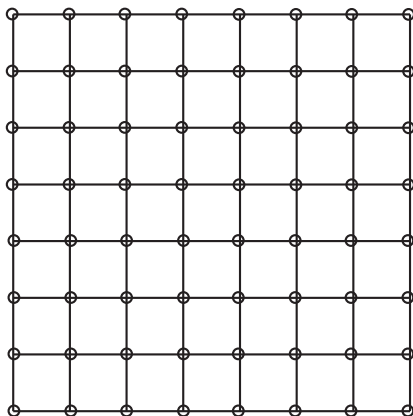


Figure 2.2: Example: Part of the square lattice  $\mathbb{Z}^2$

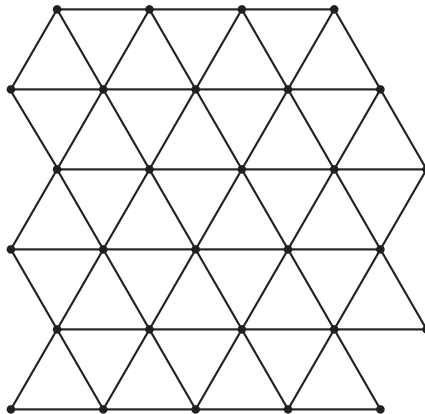


Figure 2.3: Example: Part of the triangular lattice ( $3^6$ )

sequence  $x =: x_1, x_2, \dots, x_n := y$  such that there is an edge between  $x_i$  and  $x_{i+1}$  for all  $i \in \{1, \dots, n-1\}$ . We define  $C$  to be the component of the RCM on  $X \cup \{0\}$  which contains the origin. By this we mean the set of all  $x \in X$  that are path-connected to the point at the origin. We define

$$\theta(\lambda) := P[|C| = \infty.]$$

**Proposition 2.3** *The percolation probability  $\theta$  is non-decreasing in  $\lambda$ .*

**Proof.** Suppose  $\lambda_1 < \lambda_2$ . Then consider coupled random connection models. In both models we start off with a Poisson Process  $X$  of intensity  $\lambda_2$ . We then add in edges in  $X \cup \{0\}$  according to the connection function  $f$ . We can then form a coupled random connection model with intensity  $\lambda_1$  by keeping each point of  $X$  with probability  $\frac{\lambda_1}{\lambda_2}$  independently and deleting all the others. Obviously the component containing the origin in the  $\lambda_1$  model is contained in the component containing the origin in the  $\lambda_2$  model so we get that  $\theta(\lambda_1) \leq \theta(\lambda_2)$ .  $\square$

Define the critical value

$$\lambda_f := \sup\{\lambda > 0 : \theta(\lambda) = 0\}.$$

We need a condition on the connection function  $f$  to make the model non-trivial. Namely,

$$0 < \int_{\mathbb{R}^d} f(|x|) dx < \infty. \quad (2.1)$$



This is so that all vertex degrees are almost surely finite.

**Proposition 2.4** *If condition (2.1) holds and if  $f$  is non-increasing then we have that:*

$$0 < \lambda_f < \infty.$$

**Proof.**

A proof along these lines appears in section 6.1 of [14]. We can use a branching process argument to show the lower bound. We need to find a  $\lambda > 0$  with  $\theta(\lambda) = 0$ . Let  $X_0$  be the set of points which have an edge between them and the origin. So the set  $X_0$  is a non-homogeneous Poisson process with intensity  $\lambda f(|x|)$ . We then build up the second generation as follows. Label the points of  $X_0$  by  $x_1, \dots, x_n$ . Build up the set  $X_1^1$  of points that are joined to  $x_1$  but not the origin. This is a non-homogeneous Poisson process with intensity  $\lambda(1 - f(|x|))f(|x - x_1|)$  which is independent of  $X_0$ . These points are the second generation points coming from  $x_1$ . Then build up the set of points  $X_1^2$  that are joined to  $x_2$  but not  $x_1$  or the origin. This is a non-homogeneous Poisson process with intensity  $\lambda(1 - f(|x|))(1 - f(|x - x_1|))f(|x - x_2|)$  which is independent of  $X_0$  and  $X_1^1$ . These are the second generation points coming from  $x_2$ .

Continuing this procedure in the obvious way we get  $X_1^1, \dots, X_1^n$  which makes up the complete second generation. Each Poisson process  $X_1^i$  contains the factor  $\lambda f(|x - x_i|)$  so each  $X_1^i$  can be coupled to independent non-homogeneous Poisson processes  $Y_1^i$  with intensities  $\lambda f(|x - x_i|)$ . The number of points in  $Y_1^i$  has a Poisson distribution with mean  $\lambda \int_{\mathbb{R}^d} f(|x - x_i|) dx = \lambda \int_{\mathbb{R}^d} f(|x|) dx$ . Therefore the total number of points in the second generation is bounded above by the number of points in the second generation of an ordinary Galton-Watson branching process with expected offspring equal to  $\lambda \int_{\mathbb{R}^d} f(|x|) dx$ . Iterating this procedure gives that the total number of points in the  $n$ th generation is bounded above by the number of points in the  $n$ th generation of an ordinary Galton-Watson branching process with expected offspring equal to  $\lambda \int_{\mathbb{R}^d} f(|x|) dx$ . Therefore the expected size of  $C$  satisfies:

$$E[|C|] \leq \sum_0^\infty \left( \lambda \int_{\mathbb{R}^d} f(|x|) dx \right)^n$$

This sum is finite if

$$\lambda < \left( \int_{\mathbb{R}^d} f(|x|) dx \right)^{-1}. \quad (2.2)$$

So this shows that

$$\lambda_f \geq \left( \int_{\mathbb{R}^d} f(|x|) dx \right)^{-1} > 0. \quad (2.3)$$

For the upper bound we compare the random connection model with site percolation on  $\mathbb{Z}^d$ . We let  $R = \sup\{r : f(r) > 0\}$ . We then let the site  $(a_1, \dots, a_d)$  in  $\mathbb{Z}^d$  correspond to a cube of side  $\frac{R}{4\sqrt{d}}$  centred on the point  $(\frac{Ra_1}{4\sqrt{d}}, \dots, \frac{Ra_d}{4\sqrt{d}})$ . We then build up the component from the origin as follows. We order the sites of  $\mathbb{Z}^d$  in some way  $(z_1, z_2, \dots)$  and label the corresponding cubes  $(Z_1, Z_2, \dots)$ . We build up a process with intensity  $\lambda f(|x|)$  on the cube  $Z_i$  corresponding to the first site in the ordering that is adjacent to the origin. This builds up the set of points in this cube that are connected to the origin. If at least one point appears in this process then this corresponds to the site in  $z_i$  being open, otherwise  $z_i$  is closed. We let  $X_i$  be the set of points in this process on  $Z_i$ , and these are now in  $C$ , the component containing the origin, and  $z_i$  is in  $C'$ , the component of  $\mathbb{Z}^d$  containing the origin. We now consider the first site  $z_j$  in  $\mathbb{Z}^d$  that is adjacent to  $C'$  and has not yet been declared open or closed. We build up the process with intensity  $\lambda p(x, C)$  on  $Z_j$  and if any points occur we add them to  $C$  and declare the site  $z_j$  to be open. If no points occur then we have  $z_j$  being closed. We continue and only stop if all sites adjacent to  $C'$  are closed.

Each time we build up the process in a cube  $Z_j$  we know it is adjacent to a cube which contains a point in  $C$ . As the cubes are of side  $\frac{R}{4\sqrt{d}}$  this means that any point in  $Z_j$  is within  $R/2$  of a point in  $C$ , so the intensity  $\lambda p(x, C)$  is at least  $\lambda f(R/2)$  for all  $x$  in  $Z_j$ . Therefore the probability that  $z_j$  is open is at least  $1 - \exp\left(-\lambda f(\frac{R}{2})[\frac{R}{2\sqrt{d}}]^d\right)$ . So the probability of an infinite cluster  $C'$  is strictly positive as long as  $1 - \exp\left(-\lambda f(\frac{R}{2})[\frac{R}{2\sqrt{d}}]^d\right)$  is bigger than the critical site probability for  $\mathbb{Z}^d$ . If  $C'$  is infinite then this means that  $C$  is infinite as well.

In this way we can see that the probability of an infinite component connected to the origin in the random connection model is at least as much as the probability of  $C$  being infinite in the site percolation model with percolation probability  $1 - \exp\left(-\lambda f(\frac{R}{2})[\frac{R}{2\sqrt{d}}]^d\right)$ . So if we make  $\lambda$  big enough that  $1 - \exp\left(-\lambda f(\frac{R}{2})[\frac{R}{2\sqrt{d}}]^d\right) > p_c^{site}(\mathbb{Z}^d)$  then we have percolation.  $\square$

## 2.5 Site and Bond Percolation on the Random Connection Model

The definition of  $p_c^{site}$  and  $p_c^{bond}$  can be extended to random geometric graphs. If the intensity  $\lambda$  is greater than  $\lambda_f$  then there is almost surely a unique infinite component  $D$  (as proved in section 6.4 of [14]). This is an infinite connected graph so we can perform site and bond percolation, and get  $p_c^{site}$  and  $p_c^{bond}$  as before. We define them slightly differently so that they just depend on the intensity  $\lambda$  and the connection function  $f$ . Let  $\theta(p, f, \lambda)$  be the probability that the component  $C$  that contains the point at the origin is infinite where we have a Poisson process with intensity  $\lambda$ , each point in the process being independently open with probability  $p$  and then open points being connected with probability  $f$ . So this is the site percolation model. Define  $p_c^{site} = \sup\{p : \theta(p, f, \lambda) = 0\}$ . It can easily be seen by the properties of the Poisson process that  $\theta(p, f, \lambda) = \theta(1, f, p\lambda)$ , so therefore  $p_c^{site} = \frac{\lambda_f}{\lambda}$ , so  $p_c^{site}$  is clearly in  $(0, 1)$  for  $\lambda > \lambda_f$ .

Similarly let  $\theta^*(p, f, \lambda)$  be the probability that the component  $C$  that contains the point at the origin is infinite where we have a Poisson process with intensity  $\lambda$ , and then open points being connected with probability  $pf$ . So this is the bond percolation model. We then define  $p_c^{bond} = \sup\{p : \theta^*(p, f, \lambda) = 0\}$ . This way of defining  $p_c^{site}$  and  $p_c^{bond}$  makes it clear that they are functions of  $f$  and  $\lambda$  and not random variables.

**Proposition 2.5** *The critical bond probability  $p_c^{bond}$  is in  $(0, 1)$  for  $\lambda > \lambda_f$ .*

*Proof.* For the upper bound we have that  $p_c^{bond} \leq p_c^{site} < 1$ . For the lower bound we note that  $\theta^*(p, f, \lambda) = \theta^*(1, pf, \lambda)$ . So by equation (2.3) we see that  $\theta^*(p, f, \lambda) = 0$  if  $\lambda \int_{\mathbb{R}^d} pf(x)dx < 1$ . Therefore  $p_c^{bond} \geq (\lambda \int_{\mathbb{R}^d} f(x)dx)^{-1} > 0$ .

## 2.6 Applications

One application of percolation theory could be the spread of disease. For instance an orchard of trees could be set up in a square lattice pattern and the spread of a disease from tree to tree could be modelled by percolation. Or the trees could be a natural forest and be arranged in a more random manner, maybe similar to a Poisson Process in which case the spread of disease could be modelled by a Random Connection Model. Having a decreasing connection function would seem to be a reasonable assumption in this case

as disease is more likely to spread directly between two trees that are close to each other. Site percolation on the Random Connection Model could represent a certain proportion of trees having natural immunity to the disease, or maybe one can control the proportion of trees immune to the disease by treating them with something and wishes to treat the smallest amount possible to stop large scale spread of the disease. Another application could be mobile phone coverage, where the probability of one phone mast being able to receive a signal from another phone mast depends on the distance between the two and percolation corresponds to messages being able to be transferred across the country via a network of connections between phone masts.

### 3 Random Connection Model: previously known results

This section contains some known results in continuum percolation including results that have been proved using an enhancement technique, which is widely used in this thesis. It also includes a couple of results that are used later in the thesis.

#### 3.1 Squashing and shifting transformations

Given a connection function  $f$  and  $p$  in  $(0, 1)$  define

$$S_p f(x) := pf(p^{1/d}x). \quad (3.1)$$

This function ‘squashes’ the original function but ‘spreads it out’, in such a way that the expected degree of the point at the origin is the same for the squashed function as for the original. Franceschetti et al. [7] proved that

$$\lambda_f \geq \lambda_{S_p f}$$

by making use of the inequality  $p_c^{bond} \leq p_c^{site}$ .

In the same paper they also considered a shift transformation where given a connection function  $f$  and  $s > 0$  the function  $f_s^{shift}(x) := f[c^{-1}(x - s)]I_{x > s}$  where the constant  $c$  is chosen so that the expected degree of the point at the origin remains the same. They showed that as  $s$  goes to  $\infty$  the critical degree tends to 1, where the critical degree is the expected degree of 0 at the critical intensity.

#### 3.2 Spread out Percolation model

Given an intensity  $\lambda$  and connection function  $f$  set  $\mu(\lambda, f) := \lambda \int_{\mathbb{R}^d} f(|x|)dx$  to be the expected degree of 0. Let  $\phi$  be a fixed probability density function. Consider when  $f$  is a small constant multiple of  $\phi$ , i.e. for  $h > 0$  set  $f_h(x) := h\phi(x)$ . Then  $\mu(\lambda, f_h) = \lambda h$ . With fixed  $\lambda$  define

$$\mu_c(\lambda) := \lambda \inf \{h \in [0, \|\phi\|_\infty^{-1}] : \lambda > \lambda_{f_h}\}.$$

Then Penrose [18] showed that  $\mu_c(\lambda) \rightarrow 1$  as  $\lambda \rightarrow \infty$ , and also that as  $\lambda \rightarrow \infty$  with  $\mu$  fixed, then

$$P_\lambda[|C| = \infty] \rightarrow \psi(\mu),$$

where  $\psi(\mu)$  is the survival probability of a Galton-Watson process with parameter  $\mu$ . Making the intensity very high and the probabilities of each connection smaller is just a condensed version of the squashed model above so this shows that as connections become very spread out but unreliable the model ‘tends’ to a Galton-Watson process.

### 3.3 High Dimensions

It has already been shown (see (2.3)) by comparing the cluster at the origin with a Galton-Watson survival process that the critical intensity  $\lambda_f$  is at least as much as  $(\int_{\mathbb{R}^d} f(|x|)dx)^{-1}$ . For fixed  $f$  let  $\lambda_f^d$  be the critical intensity for the random connection model in  $d$  dimensions. Let  $I_d(f) = (\int_{\mathbb{R}^d} f(|x|)dx)$  so  $\lambda I_d(f)$  is the expected degree of a point at the origin. Penrose, Meester and Sarkar [19] proved that under certain conditions on the connection function  $f$  the limit as  $d \rightarrow \infty$  satisfies:

$$\lambda_f^d I_d(f) \rightarrow 1.$$

So in high dimensions the cluster at the origin behaves more and more like a Galton-Watson process with mean  $\lambda I_d(f)$ .

### 3.4 Enhancement

The relationship between  $p_c^{site}$  and  $p_c^{bond}$  has been studied for many graphs, and in particular whether they are equal or if  $p_c^{bond} < p_c^{site}$ . Grimmett and Stacey [9] showed that it is a strict inequality for all hypercubic lattices with dimension  $d \geq 2$  by using an enhancement technique originally developed by Aizenman and Grimmett [1] and Menshikov [15]. The idea is to enhance the site percolation model (by making more sites open) in a way which strictly increases the probability of percolation but such that if this enhanced model percolates then so does the bond percolation model. They also prove the strict inequality for more general graphs. In particular if a graph  $G$  is infinite, finitely transitive, connected, locally finite and bridgeless (where a bridge is an edge that disconnects the graph if it is removed) and  $p_c^{bond}(G) \neq 1$  then  $p_c^{bond}(G) < p_c^{site}(G)$  (A graph  $G$  is finitely transitive if the group of automorphisms of  $G$  has only finitely many orbits). Random connection models do not satisfy this as they are not finitely transitive; however they have a ‘positive density’ of cycles so it seems like  $p_c^{bond} < p_c^{site}$  should hold for all of these models.

### 3.5 Shape percolation

Another use of enhancement was by Roy and Tanemura [22] in shape percolation. The shape percolation model consists of a Poisson point process with a shape centred at each point (and all shapes are the same size and orientation). Then there is an edge between two points if the shapes associated with them intersect. There is a critical intensity depending on the shape above which there is percolation and below which there is not. Jonasson [11] proved that for a convex shape of fixed area, a shape cannot have a lower critical intensity than a triangle. Roy and Tanemura used enhancement to show that the critical intensity for a triangle is strictly lower than for any other convex shape, and they also showed that if a convex shape is strictly contained in another convex shape then the critical intensity is strictly higher for the smaller shape.

### 3.6 Margulis-Russo lemma

Given a finite collection of independent identically distributed Bernoulli random variables,  $X_1, X_2, \dots, X_n$ , then we say that an event  $A$  is increasing if increasing any of the variables  $X_i$  while keeping all other variables fixed cannot stop  $A$  from occurring. Let  $\omega : \{0, 1\}^n \rightarrow \{0, 1\}$  take a collection  $X_i$  to 1 if  $A$  occurs on these values and 0 otherwise. Then  $A$  is increasing if for all  $i \in [1, n]$ ;

$$\begin{aligned} \omega(X_1, X_2, \dots, X_{i-1}, 0, X_{i+1}, \dots, X_n) &= 1 \\ \Rightarrow \omega(X_1, X_2, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n) &= 1. \end{aligned}$$

Given a set of variables  $X_1, X_2, \dots, X_n$  a variable  $X_i$  is said to be pivotal if

$$\omega(X_1, X_2, \dots, X_{i-1}, 0, X_{i+1}, \dots, X_n) = 0$$

and

$$\omega(X_1, X_2, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n) = 1,$$

i.e the event  $A$  occurs if  $X_i = 1$  but not if  $X_i = 0$ . Then the Margulis-Russo Lemma states the following (see [23], Lemma 3).

**Theorem 3.1** *Let  $A$  be an increasing event on a finite collection of independent identically distributed Bernoulli random variables,  $X_1, X_2, \dots, X_n$  with parameter  $p$ . Then*

$$\frac{dP_p[A]}{dp} = \sum_i P[X_i \text{ pivotal}]$$

### 3.7 Palm Theory

This section of theory for Poisson Processes can be found in Penrose [16].

**Theorem 3.2** *Let  $\lambda > 0$ . Suppose  $h(y, X)$  is a bounded measurable function defined on all pairs of the form  $(y, X)$  with  $X$  a finite subset of  $\mathbb{R}^d$  and  $y$  a point of  $X$ . Let  $A$  be a bounded set with Lebesgue measure  $0 < |A| < \infty$ . Let  $\mathcal{P}_\lambda(A)$  be a Poisson Process with intensity  $\lambda$  restricted to  $A$ . Then*

$$E \sum_{x \in \mathcal{P}_\lambda(A)} h(x, \mathcal{P}_\lambda(A)) = \lambda |A| E h(y, y \cup \mathcal{P}_\lambda(A))$$

where the sum on the left hand side is over all the points of the point set  $\mathcal{P}_\lambda(A)$ , and on the right-hand side the point  $y$  is a point uniformly distributed over  $A$  and independent of  $\mathcal{P}_\lambda(A)$ .

**Proof.** Conditional on  $N(\mathcal{P}_\lambda(A)) = n$  the distribution of the point set is that of a collection  $\chi_n$  of  $n$  independent points uniformly distributed on  $A$ . By conditioning on  $N(\mathcal{P}_\lambda(A)) = n$  we get

$$\begin{aligned} E \sum_{x \in \mathcal{P}_\lambda(A)} h(x, \mathcal{P}_\lambda(A)) &= \sum_{n=1}^{\infty} \frac{(\lambda |A|)^n e^{-\lambda |A|}}{n!} n E[h(y, y \cup \chi_{n-1})] \\ &= \lambda |A| \sum_{m=0}^{\infty} \frac{(\lambda |A|)^m e^{-\lambda |A|}}{m!} E[h(y, y \cup \chi_m)] \end{aligned}$$



## 4 Random Connection Model: statement of results

The results in this section show the strict inequality  $p_c^{\text{bond}} < p_c^{\text{site}}$  for Random connection model with functions subject to certain conditions, and also show strict inequalities for the squashing transformation. The first result concerns Gilbert's graph which is denoted by  $G(\mathcal{P}_\lambda, 1)$ . For this special case the quantity  $\lambda_c$  is used instead of  $\lambda_f$ .

**Theorem 4.1** *Consider  $G(\mathcal{P}_\lambda, 1)$  for  $\lambda > \lambda_c$ . We have  $p_c^{\text{site}} > p_c^{\text{bond}}$ .*

The next result generalizes Theorem 4.1, and concerns the RCM with the connection function  $f$  having bounded support.

**Theorem 4.2** *Consider  $RCM(\mathcal{P}_\lambda, f)$  for  $\lambda > \lambda_f$ . If  $f$  is non-increasing and  $0 < \sup\{a : f(a) > 0\} < \infty$  then  $p_c^{\text{site}} > p_c^{\text{bond}}$ .*

The next result concerns the RCM with the connection function  $f$  having infinite support. We impose the following condition to make sure that all vertices have finite degree almost surely.

$$0 < \int_{\mathbb{R}^d} f(|z|) dz < \infty, \quad (4.1)$$

We also impose the following condition.

$$\inf_{r>0} \left\{ \frac{f(r+1)}{f(r)} \right\} > 0 \quad (4.2)$$

**Theorem 4.3** *Consider  $RCM(\mathcal{P}_\lambda, f)$  for  $\lambda > \lambda_f$ . If  $f$  is non-increasing and satisfies conditions (4.1) and (4.2) then  $p_c^{\text{site}} > p_c^{\text{bond}}$ .*

The next result shows a strict inequality if the connection function  $f$  is replaced by a weaker connection function  $pf$  with  $p \in (0, 1)$  a constant. The weak inequality  $\lambda_{pf} \geq \lambda_f$  is clear, but the next results improves it to a strict inequality.

**Theorem 4.4** *Let  $q_0 \in (0, 1)$ . Suppose the connection function  $f$  satisfies the hypotheses of Theorem 4.2. Then  $\lambda_{q_0 f} > \lambda_f$ .*

The next result improves the weak inequality proved by Franceschetti et al to a strict inequality.

**Theorem 4.5** *Let  $p \in (0, 1)$ . Suppose the connection function  $f$  satisfies the hypotheses of Theorem 4.2. Define  $S_p f$  as in (3.1). Then  $\lambda_{S_p f} < \lambda_f$ . Also, the inequality (2.3) is strict.*

These results are also shown for the infinite support case.

**Theorem 4.6** *Let  $q_0 \in (0, 1)$ . Suppose the connection function  $f$  satisfies the hypotheses of Theorem 4.3. Then  $\lambda_{q_0 f} > \lambda_f$ .*

**Theorem 4.7** *Let  $p \in (0, 1)$ . Suppose the connection function  $f$  satisfies the hypotheses of Theorem 4.3. Define  $S_p f$  as in (3.1). Then  $\lambda_{S_p f} < \lambda_f$ . Also, the inequality (2.3) is strict.*

## 5 Gilbert's Graph: Proof of Theorem 4.1

This section contains the proof of Theorem 4.1 using the enhancement technique. Throughout this section we consider Gilbert's graph  $G(\mathcal{P}_\lambda, 1)$  with  $d = 2$  and with  $\lambda > \lambda_c$ . We now describe the enhancement. The objective is to describe a way to add open vertices to the site percolation model to make the probability of an infinite cluster bigger, without changing the bond percolation model. To do so, we introduce two kinds of coloured vertices, red vertices (the original open vertices) and green vertices (closed vertices which have been enhanced) and for any two vertices  $x, y$  we write that  $x \sim y$  if they are joined by an edge. In  $G(\mathcal{P}_\lambda, 1)$ , if we have vertices  $v, w, x, y, z$  such that  $x$  is closed, has no neighbours other than  $v, w, y, z$ , which are all red, and  $v \sim w$  and  $y \sim z$  but there are no other edges amongst  $v, w, y$  and  $z$  then we say  $x$  is *correctly configured* in  $G(\mathcal{P}_\lambda, 1)$ , and refer to this as a *bow tie* configuration of edges. If a vertex  $x$  is correctly configured we make it green with probability  $q$ , independently of everything else; see Figure 5.1.

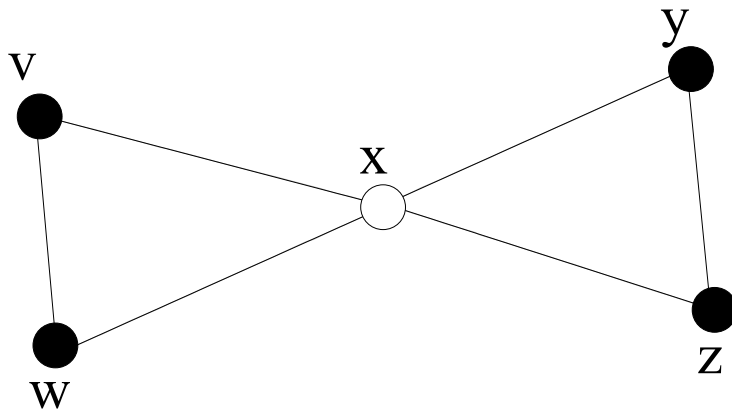


Figure 5.1: The bow tie enhancement.

Let  $B_n$  be the open disc of radius  $n$  centred at the origin. Let  $\underline{Y} = (Y_i, i \geq 0)$  and  $\underline{Z} = (Z_i, i \geq 0)$  be sequences of independent uniform  $[0, 1]$  random variables. List the vertices of  $\mathcal{P}_\lambda$  in order of increasing distance from the origin as  $x_1, x_2, x_3, \dots$ . Declare a vertex  $x_i$  to be *red* if  $Y_i < p$  and *closed* otherwise. Once the sets of red and closed vertices have been decided in this way, apply the enhancement by declaring each closed vertex  $x_j$  to be *green*

if it is correctly configured and  $Z_j < q$ . We shall sometimes need to consider the Poisson process with an extra vertex inserted at  $x \in B_n$ , in which case the extra vertex has values  $Y_0$  and  $Z_0$  associated with it. We shall refer to vertices that are either red or green as being *coloured*.

Let  $\partial B_n$  be the annulus  $B_n \setminus B_{n-0.5}$  and let  $A_n$  be the event that for the Poisson process  $\mathcal{P}_\lambda \cap B_n$ , there is a path from a coloured vertex in  $B_{0.5}$  to a coloured vertex in  $\partial B_n$  in  $G(\mathcal{P}_\lambda, 1) \cap B_n$  using only coloured vertices (note that  $A_n$  is based on a process completely inside  $B_n$ ; we do not allow vertices outside of  $B_n$  to affect possible enhancements inside  $B_n$ ). For  $x \in B_n$ , let  $A_n^x$  be defined the same way as  $A_n$ , but in terms of the point process  $(\mathcal{P}_\lambda \cap B_n) \cup \{x\}$ , i.e. the Poisson process in  $B_n$  with a point inserted at  $x$ .

Let  $\theta_n(p, q)$  be the probability that  $A_n$  occurs, and define

$$\theta(p, q) \equiv \liminf_{n \rightarrow \infty} (\theta_n(p, q)).$$

The following proposition states that  $\theta(p, q)$  is indeed the percolation function associated to the enhanced model. From now on we use ‘vertex’ to refer to a point of the Poisson process and ‘point’ to refer to an arbitrary location in  $\mathbb{R}^2$ .

**Proposition 5.1** *There is a.s. an infinite connected component in  $G(\mathcal{P}_\lambda, 1)$  using only red and green vertices if and only if  $\theta(p, q) > 0$ .*

**Proof of Proposition 5.1.** For the if part let  $A'_n$  be the event that there is a coloured path from  $B_{0.5}$  to outside  $B_{n-2}$ , so  $A_n$  is contained in  $A'_n$ . Let  $\phi_n(p, q)$  be the probability of  $A'_n$  occurring (which is monotone in  $n$ ), and let  $\phi(p, q)$  be the limit as  $n$  goes to  $\infty$ . Therefore  $\phi_n(p, q) \geq \theta_n(p, q)$  for all  $n$  so  $\phi(p, q) \geq \theta(p, q) > 0$ , but  $\phi(p, q)$  is just the probability of there being an infinite coloured component intersecting  $B_{0.5}$  and it is well known that there is almost surely an infinite coloured component if  $\phi(p, q) > 0$ .

For the only if part, if there is almost surely an infinite component then  $\phi(p, q) > 0$ . Given  $n \geq 6$ , we build up the Poisson process on the whole of  $B_{n-3}$ . If there are any closed vertices that are not definitely correctly or incorrectly configured, we build up the process in the rest of their 1-neighbourhood, and this determines whether they are green or uncoloured. If any more closed vertices occur they cannot be correctly configured as they will be joined to a closed vertex. Therefore we have built up the process everywhere in a region  $R$  with  $B_{n-3} \subset R \subset B_{n-2}$ , and all uncoloured vertices

at this stage will remain uncoloured. Let  $V$  be the set of coloured vertices that are joined by a coloured path to a coloured vertex in  $B_{0.5}$  at this stage.

Next, we build out the process radially symmetrically from  $B_{n-3}$  (apart from where the process has already been built up) until a vertex  $v$  occurs that is connected to a vertex in  $V$ . Let  $J$  be the event that such a vertex  $v$  occurs, so  $J$  must occur for  $A'_n$  to occur. Assuming  $J$  occurs, set  $r = |v|$ , so  $r \in [n-3, n-1)$ . Then we can find points  $a_1, a_2, \dots, a_9$  on the line  $0v$  extended away from the origin such that  $a_1$  is  $r+0.3$  from the origin,  $a_2$  is  $r+0.6$  from the origin and so on. Surround  $a_1, \dots, a_9$  with circles  $D_1, \dots, D_9$  of radius 0.05 around them. If there is at least one red vertex in each one of these little circles that is contained in  $B_n$  when the process continues to the whole of  $B_n$ , and  $v$  is also red then  $A_n$  occurs. Therefore if  $J$  occurs then the conditional probability of  $A_n$  occurring is at least  $\gamma$ , where

$$\gamma = p(1 - \exp(-0.0025\lambda p\pi))^9,$$

as this is the probability of getting at least one red vertex in each little circle and  $v$  being red. Therefore  $\theta_n(p, q) \geq \gamma P[J] \geq \gamma \phi(p, q)$  for all  $n \geq 6$ , so  $\theta(p, q) \geq \gamma \phi(p, q) > 0$ .  $\square$

Our next lemma provides an analogue of the Margulis-Russo formula for the enhanced continuum model. First, we need to introduce the notion of pivotal vertices.

Given the configuration  $(\mathcal{P}_\lambda, \underline{Y}, \underline{Z})$  and inserting a vertex at  $x$  we say that  $x$  is *1-pivotal* in  $B_n$  if putting  $Y_0 = 0$  means that  $A_n^x$  occurs but putting  $Y_0 = 1$  means it does not. Notice that  $x$  can either complete a path (but it cannot do via being enhanced), or it could make another closed vertex correctly configured which in turn would complete a path. We say that  $x$  is *2-pivotal* in  $B_n$  if inserting a vertex at  $x$  and putting  $Z_0 = 0$  means  $A_n^x$  occurs but putting  $Z_0 = 1$  means it does not. That is,  $Y_0 > p$  and adding a closed vertex  $v$  at  $x$  means  $v$  is correctly configured and enhancing it to a green vertex means  $A_n^x$  occurs but otherwise it does not.

For  $i = 1, 2$  let  $E_{n,i}(x)$  be the event that  $x$  is  $i$ -pivotal in  $B_n$ , and set  $P_{n,i}(x, p, q) := P[E_{n,i}(x)]$ .

**Lemma 5.1** *For all  $n > 0.5$  and  $p \in (0, 1)$  and  $q \in (0, 1)$  it is the case that*

$$\frac{\partial \theta_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,1}(x, p, q) dx \quad (5.1)$$

and

$$\frac{\partial \theta_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,2}(x, p, q) dx. \quad (5.2)$$

**Proof.** Let  $\mathcal{F}$  be the  $\sigma$ -algebra generated by the locations but not the colours of the vertices of  $\mathcal{P}_\lambda \cap B_n$ . Let  $N_1$  be the number of 1-pivotal vertices. Define  $\mathcal{F}$ -measurable random variables,  $X_{p,q}$  and  $Y_{p,q}$  as follows;  $X_{p,q}$  is the conditional probability that  $A_n$  occurs, and  $Y_{p,q}$  is the conditional expectation of  $N_1$ , given the configuration of  $\mathcal{P}_\lambda$ . By the standard version of the Margulis-Russo formula for an increasing event defined on a finite collection of Bernoulli variables (see Theorem 3.1 ),

$$\lim_{h \rightarrow 0} h^{-1}(X_{p+h,q} - X_{p,q}) = Y_{p,q}, \quad a.s.$$

Let  $M$  denote the total number of vertices of  $\mathcal{P}_\lambda$  in  $B_n$ . By the standard coupling of Bernoulli variables, and Boole's inequality (subadditivity of measures),  $|X_{p+h,q} - X_{p,q}| \leq |h|M$  almost surely, and since  $M$  is integrable we have by dominated convergence that

$$\frac{\partial \theta_n(p, q)}{\partial p} = \lim_{h \rightarrow 0} E[h^{-1}(X_{p+h,q} - X_{p,q})] = E[Y_{p,q}] = E[N_1], \quad (5.3)$$

and by a standard application of the Palm theory of Poisson processes (see Theorem 3.2), the right hand side of (5.3) equals the right hand side of (5.1).

The proof of (5.2) is similar. Let  $N_2$  be the number of 2-pivotal vertices. Let  $X_{p,q}$  be as before and define the  $\mathcal{F}$ -measurable random variable  $Z_{p,q}$  to be the conditional expectation of  $N_2$ , given the configuration of  $\mathcal{P}_\lambda$ . Then,

$$\lim_{h \rightarrow 0} h^{-1}(X_{p,q+h} - X_{p,q}) = Z_{p,q}, \quad a.s.$$

Again,  $|X_{p,q+h} - X_{p,q}| \leq |h|M$  almost surely, and since  $M$  is integrable we have by dominated convergence that

$$\frac{\partial \theta_n(p, q)}{\partial q} = \lim_{h \rightarrow 0} E[h^{-1}(X_{p,q+h} - X_{p,q})] = E[Z_{p,q}] = E[N_2], \quad (5.4)$$

and by a standard application of the Palm theory of Poisson processes (see Theorem 3.2), the right hand side of (5.4) equals the right hand side of (5.2).  $\square$

## 5.1 The Key Result

The key step in proving Theorem 4.1 is given by the following result.

**Lemma 5.2** *There is a continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n > 100$ ,  $x \in B_n$  and  $(p, q) \in (0, 1)^2$ , we have*

$$P_{n,2}(x, p, q) \geq \delta(p, q)P_{n,1}(x, p, q). \quad (5.5)$$

The idea behind the proof is to have a circle  $C_r$  of fixed radius around a point  $x$  and then carry out the process outside this circle. Then for  $x$  to be 1-pivotal there must be a coloured path from  $\partial B_n$  to near the edge of  $C_r$  and also a coloured path from  $B_{0.5}$  to near the edge of  $C_r$ , but there cannot be a coloured path from  $B_{0.5}$  to  $\partial B_n$ . If all this occurs then when we carry out the process inside  $C_r$  one would expect there to be a chance of a configuration that makes  $x$  2-pivotal, which is independent of  $n$  and  $x$ . However it is not straightforward to prove as the 2 paths could only just be in reach of  $C_r$  or could ‘interfere’ with each other, making it hard to find a point in  $C_r$  that can connect to one path but not the other.

Before proving this, we give a result saying that we can assume there are only red vertices inside an annulus of fixed size. For  $x \in \mathbb{R}^2$ , and  $0 \leq \alpha < \beta$ , let  $C_\alpha(x)$  be the closed circle (i.e., disc) of radius  $\alpha$  centred at  $x$ , and let  $A_{\alpha,\beta}(x)$  denote the annulus  $C_\beta(x) \setminus C_\alpha(x)$ . Given  $n$  and given  $x \in B_n$ , let  $R_n(x, \alpha, \beta)$  be the event that all vertices in  $A_{\alpha,\beta}(x) \cap B_n$  are red.

**Lemma 5.3** *Fix  $\alpha > 3$  and  $\beta > \alpha + 3$ . There exists a strictly positive continuous function  $\delta_1 : (0, 1)^2 \rightarrow (0, \infty)$ , such that for all  $(p, q) \in (0, 1)^2$ , all  $n > \beta + 3$  and all  $x \in B_n$  with  $|x| < \alpha - 2$  or  $|x| > \beta + 2$ , we have*

$$P[E_{n,1}(x) \cap R_n(x, \alpha, \beta)] \geq \delta_1(p, q)P_{n,1}(x).$$

**Proof.** We shall consider a modified model, which is the same as the enhanced model but with enhancements suppressed for all those vertices lying in  $A_{\alpha-1,\beta+1} := A_{\alpha-1,\beta+1}(x)$ . Let  $E'_{n,1}(x)$  be the event that  $x$  is 1-pivotal in the modified model.

Returning to the original model, we first create the Poisson process of intensity  $\lambda$  in  $B_n \setminus A_{\alpha-1,\beta+1}$ , and determine which of these vertices are red. Then we build up the Poisson process of intensity  $\lambda$  inside  $B_n \cap A_{\alpha-1,\beta+1}$  and for all of these new vertices with more than 4 neighbours, or with at least one closed neighbour outside  $A_{\alpha-1,\beta+1}$ , we decide whether they are red or

closed. This decides whether or not they are coloured as these vertices cannot possibly become green because they are not correctly configured. We now can tell which of the closed vertices outside  $A_{\alpha-1,\beta+1}$  are correctly configured, and we determine which of these are green.

This leaves a set  $W$  of vertices inside  $A_{\alpha-1,\beta+1}$  that have at most four neighbours. If we surround each vertex in  $W$  by a circle of radius 0.5 then we cannot have any point covered by more than 5 of these circles as this means that there is a vertex in  $W$  with at least 5 neighbours. All of these circles are contained in  $C_{\beta+2}$ , which has area  $\pi(\beta+2)^2$ . Therefore

$$|W| \leq \frac{5\pi(\beta+2)^2}{0.5^2\pi} = 20(\beta+2)^2. \quad (5.6)$$

For  $x$  to have any possibility of being 1-pivotal, at this stage there must be a set  $W'$  contained in  $W$  such that if every vertex in  $W'$  is coloured and every vertex in  $W \setminus W'$  is uncoloured then  $x$  becomes 1-pivotal. In this case, with probability at least  $[p(1-p)]^{20(\beta+2)^2}$  we have every vertex in  $W'$  red and every vertex in  $W \setminus W'$  closed, which would imply event  $E'_{n,1}(x)$  occurring. Therefore  $P[E'_{n,1}(x)] \geq [p(1-p)]^{20(\beta+2)^2} P[E_{n,1}(x)]$ .

Now we note that the occurrence or otherwise of  $E'_{n,1}(x)$  is unaffected by the addition or removal of closed vertices in  $A_{\alpha,\beta}(x)$ . This is because the suppression of enhancements in  $A_{\alpha-1,\beta+1}$  means that these added or removed vertices cannot be enhanced themselves, and moreover any vertices they cause to be correctly or incorrectly configured also cannot be enhanced.

Consider creating the marked Poisson process in  $B_n$ , with each Poisson point (vertex)  $x_i$  marked with the pair  $(Y_i, Z_i)$ , in two stages. First, add all marked vertices in  $B_n \setminus A_{\alpha,\beta}(x)$ , and just the red vertices in  $B_n \cap A_{\alpha,\beta}(x)$ . Secondly, add the closed vertices in  $B_n \cap A_{\alpha,\beta}(x)$ . The vertices added at the second stage have no bearing on the event  $E'_{n,1}(x)$ , so  $E'_{n,1}(x)$  is independent of the event that no vertices at all are added in the second stage. Hence,

$$P[E'_{n,1}(x) \cap R_n(x, \alpha, \beta)] \geq \exp(-(1-p)\lambda(\beta^2 - \alpha^2)) P[E'_{n,1}(x)],$$

with equality if  $|x| \leq n - \beta$ .

Finally, we use a similar argument to the initial argument in this proof. Suppose  $E'_{n,1}(x) \cap R_n(x, \alpha, \beta)$  occurs. Then there exist at most  $20(\beta+2)^2$  vertices in  $A_{\beta,\beta+1}(x) \cup A_{\alpha-1,\alpha}(x)$  which are correctly configured for which the possibility of enhancement has been suppressed. If we now allow these to be possibly enhanced, there is a probability of at least  $(1-q)^{20(\beta+2)^2}$  that none



of them is enhanced, in which case the set of coloured vertices is the same for the modified model as for the un-modified model and therefore  $E_{n,1}(x)$  occurs. Taking

$$\delta_1(p, q) = [p(1-p)(1-q)]^{20(\beta+2)^2} \exp(-(1-p)\lambda(\beta^2 - \alpha^2)),$$

we are done.  $\square$

**Proof of Lemma 5.2.** Fix  $p$  and  $q$ . Also fix  $n$  and  $x \in B_n$ , and just write  $P_{n,i}(x)$  for  $P_{n,i}(x, p, q)$ . Define event  $E_{n,1}(x)$  as before, so that  $P_{n,1}(x) = P[E_{n,1}(x)]$ . Also, for  $0 < r < s$  write  $C_r$  for the disc  $C_r(x)$  and  $A_{r,s}$  for the annulus  $A_{r,s}(x)$ . For now we assume  $30.5 < |x| < n - 30.5$ . We create the Poisson process of intensity  $\lambda$  everywhere on  $B_n$  except inside  $C_{30}$ , and decide which of these vertices are red.

Now we create the process of only the red vertices in  $A_{25,30}$  (a Poisson process of intensity  $p\lambda$  in this region). Assuming there will be no closed vertices in  $A_{25,30}$ , we then know which of the closed vertices outside  $C_{30}$  are correctly configured, and we determine which of these are green.

Having done all this, let  $V$  denote the set of current vertices in  $B_n \setminus C_{25}$  that are connected to  $B_{0.5}$  at this stage (by connected we mean connected via a coloured path), and let  $T$  denote the set of current vertices in  $B_n \setminus C_{25}$  that are connected to  $\partial B_n$ .

Let  $N(V)$  be the 1-neighbourhood of  $V$  and let  $N(T)$  be the 1-neighbourhood of  $T$ . Recalling that  $A \triangle B := (A \cup B) \setminus A \cap B$ , we build up the red process inwards (i.e., towards  $x$  from the boundary of  $C_{25}$ ) on  $C_{25} \cap (N(V) \triangle N(T))$  until a red vertex  $y$  occurs (if such a vertex occurs). Set  $r = |y - x|$ . Suppose  $y \in N(V)$  (if instead  $y \in N(T)$  we would reverse the roles of  $V$  and  $T$  in the sequel). Then if  $T \cap C_{r+0.05} \neq \emptyset$  we say that event  $F$  has occurred and we let  $z$  denote an arbitrarily chosen vertex of  $T \cap C_{r+0.05}$ . Otherwise, we build up the red process inwards on  $C_r \cap N(T) \setminus N(V)$  until a red vertex  $z$  occurs (if such a vertex occurs).

Let  $E_2$  be the event that (i) such vertices  $y$  and  $z$  occur, and (ii) the sets  $V$  and  $T$  are disjoint, and (iii)  $|y - z| > 1$ , and (iv) there is no path from  $y$  to  $z$  through coloured vertices in  $B_n \setminus C_{25}$  that are not in  $V \cup T$ . If  $E_{n,1}(x) \cap R_n(x, 20, 30)$  occurs, then  $E_2$  must occur.

Now suppose  $E_2 \cap F$  has occurred. Let  $a$  be the point (again we use ‘point’ to refer to a point in  $\mathbb{R}^2$ ) which is at distance  $r$  from  $x$  and distance 1 from  $y$  on the opposite side of the line  $xy$  to the side  $z$  is on (see Figure

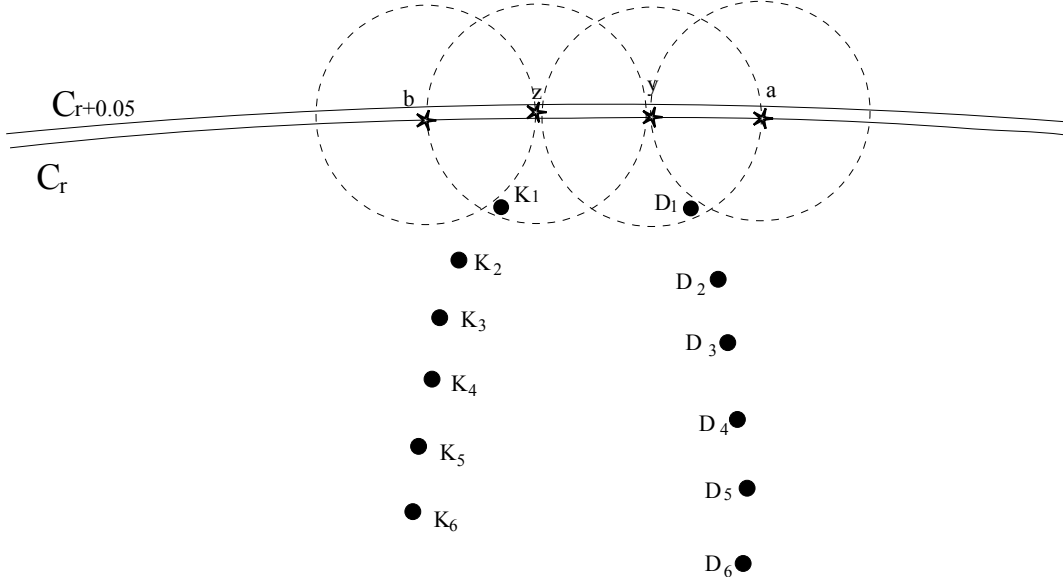


Figure 5.2: Our convention in the diagrams is to indicate points with lower case letters, and areas with upper case letters. The dashed circles are of radius 1. Here the event  $F$  occurs.

5.2). Similarly let  $b$  be the point lying at distance 1 from  $z$  and distance  $r$  from  $x$ , on the opposite side of  $xz$  to  $y$ .

Let  $a_1$  be the point lying inside  $C_r$  at distance 1.01 from  $a$  and 0.99 from  $y$ , and let  $D_1$  be the disc  $C_{0.005}(a_1)$ . Let  $b_1$  be the point at distance 1.01 from  $b$  and 0.99 from  $z$ , and let  $K_1 := C_{0.005}(b_1)$ .

Any red vertex in  $D_1$  will be connected to  $y$  (and therefore to a path to  $B_{0.5}$ ) but cannot be connected to any coloured path to  $\partial B_n$  as  $a$  is the nearest place for such a vertex to be, given  $E_2 \cap F$  occurs. Any red vertex in  $K_1$  will be connected to  $z$  (and therefore a path to  $\partial B_n$ ), but not a path to  $B_{0.5}$ . Also, any vertex in  $D_1$  will be at least 1.1 away from any vertex in  $K_1$ .

Now let  $l$  be the line through  $x$  such that  $a_1$  and  $b_1$  are on different sides of the line and at equal distance from the line. We can pick points  $a_2, a_3, \dots, a_{30}$  such that  $|a_i - a_{i-1}| \leq 0.9$  for  $2 \leq i \leq 30$ , and  $\max(|a_{30} - x|, |a_{29} - x|) \leq 0.9$ , but  $|a_i - x| > 1.1$  for  $i \leq 28$ , and none of the  $a_i : i \geq 2$  are within 1 of  $C_r$  or within 0.51 of  $l$  or within 0.01 of another  $a_j$ .

Do the same on the other side of  $l$  with  $b_2, b_3, \dots, b_{30}$ . For  $2 \leq i \leq 30$ , define discs  $D_i := C_{0.005}(a_i)$  and  $K_i := C_{0.005}(b_i)$ .

Let  $I$  be the event that there is at exactly one red vertex in each of the circles  $D_i$  and  $K_i$ ,  $1 \leq i \leq 30$ , and there are no more new vertices anywhere else in  $C_{25}$ , and no closed vertices in  $C_{30} \setminus C_{25}$ . Then

$$P[I|E_2 \cap F] \geq (0.005^2 \pi \lambda p)^{60} \exp(-900\pi\lambda) =: \delta_2.$$

If the events  $E_2$ ,  $F$ ,  $I$  occur and  $Y_0 > p$  then  $x$  is 2-pivotal.

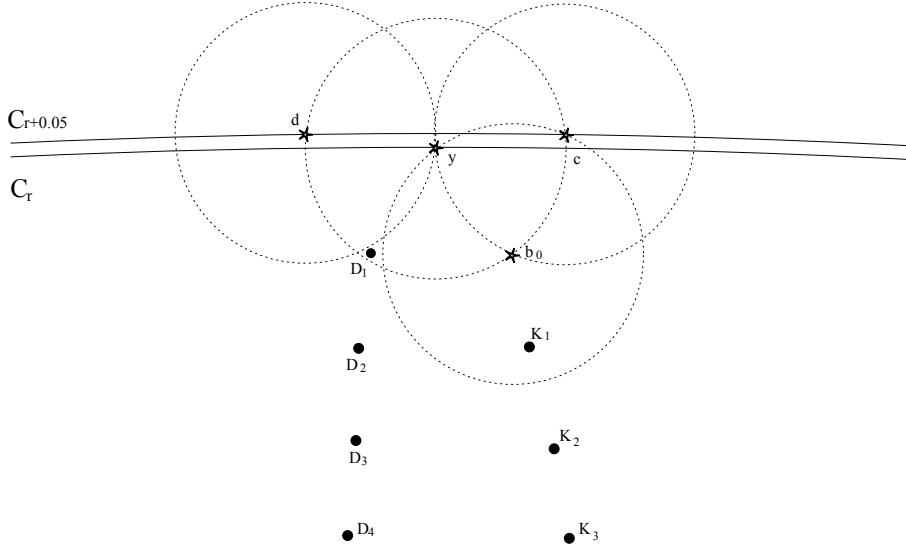


Figure 5.3: The case where  $F$  does not occur. Here  $b_0$  is the ‘worst possible’ location for  $z$ .

Now we consider the case where  $E_2$  occurs but  $F$  does not, so  $z$  is inside  $C_r$  and is connected to a vertex  $z_1$  in  $T$  that must be outside  $C_{r+0.05}$  because  $T \cap C_{r+0.05} = \emptyset$  (see Figure 5.3).

Let  $c$  be the point at distance 1 from  $y$  and  $r + 0.05$  from  $x$ , on the same side of the line  $xy$  as  $z$  (assume without loss of generality this is to the right of  $y$ ). This is the closest  $z_1$  can be. Let  $b_0$  be the point inside  $C_r$  at distance 1 from  $y$  and 1 from  $c$ , so this is the furthest left that  $z$  can be. Let  $d$  be the point at distance  $r + 0.05$  from  $x$  and 1 from  $y$ , on the other side of  $y$  to  $c$ . Let  $a_1$  be the point inside  $C_r$  at distance 1.01 from  $d$  and 0.99 from  $y$ , and let  $D_1 := C_{0.005}(a_1)$ . Then any vertex in  $D_1$  is distant at least 1.01 from  $b_0$ , and therefore from  $z$ , as  $z$  cannot be any nearer than  $b_0$ . Also any vertex in  $D_1$  will be at least 1.005 from any other vertices in  $T$ , as  $d$  is the nearest place such a point can be. As before we can then have small discs  $D_2, \dots, D_{30}$  and

$K_1, \dots, K_{30}$  (of radius 0.005) such that having one red vertex in each of these vertices ensures that  $x$  is 2-pivotal.

Given  $E_2 \setminus F$  occurs, the probability of getting 1 red vertex in each of the discs  $D_i$  and  $K_i$  for  $1 \leq i \leq 30$ , and no other new vertices in  $C_{25}$ , and no closed vertices in  $C_{30} \setminus C_{25}$ , is at least  $\delta_2$ . If this happens and also  $Y_0 > p$  then  $x$  is 2-pivotal.

So by Lemma 5.3, the probability that  $x$  is 2-pivotal satisfies

$$\begin{aligned} P_{n,2}(x) &\geq \delta_2(1-p)P[E_2 \cap F] + \delta_2(1-p)P[E_2 \cap F^c] \\ &\geq \delta_2(1-p)P[E_{n,1}(x) \cap R_n(x, 20, 30)] \\ &\geq \delta_1\delta_2(1-p)P_{n,1}(x). \end{aligned}$$

This proves the claim (5.5) for the case with  $30.5 < |x| < n - 30.5$ .

Now suppose  $|x| \leq 30.5$ . Create the Poisson process in  $B_n \setminus C_{40}$ , and decide which of these vertices are red. Then create the red process in  $A_{39,40}(x)$ , and determine which vertices in  $B_n \setminus C_{40}$  are green, assuming there are no closed vertices in  $A_{39,40}(x)$ . Then build up the red process in  $C_{39}$  inwards towards  $x$  until a vertex  $y$  occurs in the process which is connected to  $\partial B_n$ . Let  $H_1$  be the event that such a vertex  $y$  appears at distance  $r$  between 38 and 39 from  $x$ , so  $H_1$  must occur for  $E_{n,1}(x) \cap R_n(x, 20, 40)$  to occur.

If  $x$  is inside  $B_{0.5}$  we can choose points  $a_0$  and  $a_1$  such that they are both outside  $B_{0.5}$ , at distance between 0.8 and 0.9 from  $x$  and at distance between 0.1 and 0.2 from each other. We can then choose  $b_0$  and  $b_1$  such that they are both within 0.9 of  $x$ , further than 1.5 from  $a_0$  and  $a_1$  and between 0.1 and 0.2 from each other. We can then choose points  $a_2, a_3, \dots, a_{100}$  such that  $|a_i - a_{i-1}| \leq 0.9$  for  $2 \leq i \leq 100$ , and  $|a_{100} - y| \leq 0.9$ , no two  $a_i$  are within 0.1 of each other, and no  $a_i$  is within 1.1 of  $x$ ,  $b_0$  or  $b_1$ , or inside  $B_{0.5}$  for  $i \geq 2$ .

Define discs  $D_i = C_{0.05}(a_i)$  and  $K_j = C_{0.05}(b_j)$ . If there is at least one red vertex in each of these discs and no vertices anywhere else in  $C_r$ , and  $Y_0 > p$ , then  $x$  is 2-pivotal. If  $x$  is outside  $B_{0.5}$  we choose points in a similar way but make sure  $b_1$  connects with a path to  $B_{0.5}$ , using little discs  $K_2, K_3, \dots, K_{50}$  which are again of radius 0.05 and are at least 1.1 from the  $a_i$ . Therefore, setting

$$\delta_3 := (1-p)(0.05^2\pi\lambda p)^{152} \exp(-1600\pi\lambda)$$

and using Lemma 5.3, we have for some strictly positive continuous  $\delta_4(p, q)$  that

$$P_{n,2}(x) \geq \delta_3 P[H_1] \geq \delta_3 P[E_{n,1}(x) \cap R_n(x, 20, 40)] \geq \delta_3 \delta_4 P_{n,1}(x).$$

Now suppose  $|x| \geq n - 30.5$ . In this case the proof is similar. Again, create the Poisson process in  $B_n \setminus C_{40}$ . Then create the red process in  $A_{39,40}(x)$  and determine the colours of the vertices in  $B_n \setminus C_{40}$ , assuming there are no closed vertices in  $A_{39,40}(x)$ . Then build the red process in  $C_{39} \cap B_{n-0.5}$  inwards towards  $x$  until a vertex  $y$  occurs that is connected to a path of coloured vertices to  $B_{0.5}$  but not to  $\partial B_n$ . Let  $H_2$  be the event that such a vertex  $y$  occurs at distance  $r$  between 38 and 39 from  $x$ , and that there is no current coloured path from  $B_{0.5}$  to  $\partial B_n$ , so  $H_2$  must occur for  $E_{n,1}(x) \cap R_n(x, 20, 40)$  to occur. Given this vertex  $y$  we can find discs  $D_1, D_2, \dots, D_{100}$  and  $K_1, K_2, \dots, K_{50}$  of radius 0.05 as before such that having a red vertex in each of these discs but no other vertices in  $C_r$  or  $\partial B_n \cap C_{40}$ , and having  $Y_0 > p$ , ensures  $x$  is 2-pivotal. Therefore in this case

$$P_{n,2}(x) \geq \delta_3 P[H_2] \geq \delta_3 P[E_{n,1}(x) \cap R_n(x, 20, 40)] \geq \delta_3 \delta_4 P_{n,1}(x).$$

Take  $\delta(p, q) := \min(\delta_1 \delta_2 (1 - p), \delta_3 \delta_4)$ . By its construction  $\delta$  is strictly positive and continuous in  $p$  and  $q$ , and (5.5) holds for all  $x \in B_n$ , completing the proof of the lemma.  $\square$

The following proposition follows immediately from Lemmas 5.1 and 5.2.

**Proposition 5.2** *There is a strictly positive continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n \geq 100$  and  $(p, q) \in (0, 1)^2$ , we have*

$$\frac{\partial \theta_n(p, q)}{\partial q} \geq \delta(p, q) \frac{\partial \theta_n(p, q)}{\partial p}.$$

**Proof of Theorem 4.1.** Set  $p^* = p_c^{\text{site}}$  and  $q^* = (1/8)(p^*)^2$ . Then using Proposition 5.2 and looking at a small box around  $(p^*, q^*)$ , we can find  $\varepsilon \in (0, \min(p^*/2, 1 - p^*))$  and  $\kappa \in (0, q^*)$  such that for all  $n > 100$  we have

$$\theta_n(p^* + \varepsilon, q^* - \kappa) \leq \theta_n(p^* - \varepsilon, q^* + \kappa).$$

Taking the limit inferior as  $n \rightarrow \infty$ , since  $\theta$  is monotone in  $q$  we get

$$0 < \theta(p^* + \varepsilon, 0) \leq \theta(p^* + \varepsilon, q^* - \kappa) \leq \theta(p^* - \varepsilon, q^* + \kappa).$$

Now set  $p = p^* - \varepsilon$ . Then  $q^* + \kappa \leq p^2$ , so that  $\theta(p, p^2) > 0$ , and by Proposition 5.1, the enhanced model with parameters  $(p, p^2)$  percolates, i.e. has an infinite coloured component, almost surely.

We finish the proof with a coupling argument along the lines of Grimmett and Stacey [9]. Let  $E$  be the set of edges and  $V$  be the set of vertices of  $\mathcal{C}$  (the infinite component). Let  $(X_e : e \in E)$  and  $(Z_v : v \in V)$  be collections of independent Bernoulli random variables with mean  $p$ . From these we construct a new collection  $(Y_v : v \in V)$  which constitutes a site percolation process on  $\mathcal{C}$ . Let  $e_0, e_1, \dots$  be an enumeration of the edges of  $\mathcal{C}$  and  $v_0, v_1, \dots$  an enumeration of the vertices. Suppose at some point we have defined  $(Y_v : v \in W)$  for some subset  $W$  of  $V$ . Let  $\mathcal{Y}$  be the set of vertices not in  $W$  which are adjacent to some currently active vertex (i.e. a vertex  $u \in W$  with  $Y_u = 1$ ). If  $\mathcal{Y} = \emptyset$  then let  $y$  be the first vertex not in  $W$  and set  $Y_y = Z_y$  and add  $y$  to  $W$ . If  $\mathcal{Y} \neq \emptyset$ , we let  $y$  be the first vertex in  $\mathcal{Y}$  and let  $y'$  be the first currently active vertex adjacent to it, then set  $Y_y = X_{yy'}$  and add  $y$  to  $W$ . Repeating this process builds up the entire red site percolation process.

For any correctly configured vertex  $x$  with  $v, w, y, z$  as in Figure 5.1,  $x$  itself is not red. Therefore at most one edge to  $x$  has been examined, so we can find a first unexamined edge (in the enumeration) to  $v$  or  $w$ , and then to  $y$  or  $z$ . We then declare  $x$  to be green only if both of these edges are open, which happens with probability  $p^2$ . This completes the enhanced site process with  $q = p^2$  and every component in this is contained in a component for the bond process  $\{X_e\}$ .

Therefore, since the enhanced  $(p, p^2)$  site process percolates almost surely, so does the bond process, so  $p_c^{\text{bond}} \leq p < p_c^{\text{site}}$ .  $\square$

## 6 Random Connection Model: Proof of Theorem 4.2

This chapter generalises the result from the previous chapter to the random connection model for a wide range of connection functions with bounded support. The strategy behind the proof follows that of the proof for Gilbert's graph and uses lemmas equivalent to Lemmas 5.1, 5.3 and 5.2. The proof of Lemma 5.2 needs quite a bit of extra technical work to be adapted to this case. Throughout the section we take  $d = 2$  but the method can be adapted to higher dimensions.

### 6.1 RCM: the key lemma

This section is devoted to stating and proving Lemma 6.1 below, which is the key step in subsequently proving Theorems 4.2 and 4.4. We consider the RCM with connection function  $f : [0, \infty) \rightarrow [0, 1]$ . Throughout this section we assume that  $f$  is non-increasing and, moreover, that

$$\sup\{a : f(a) > 0\} = 1. \quad (6.1)$$

Fix  $x \in \mathbb{R}^2$  and (as in the preceding section) for  $r < s$  let  $C_r$  denote the disc of radius  $r$  centred at  $x$  and let  $A_{r,s}$  denote the annulus  $C_s \setminus C_r$ .

We consider the RCM on a Poisson process in  $C_{29}$ , under certain *boundary conditions*, represented by three finite disjoint sets  $V, T$  and  $S$  in  $\mathbb{R}^2 \setminus C_{29}$ , together with a collection  $\mathcal{E}$  of edges amongst the vertices (i.e., elements) of  $S$ . We write  $\mathbf{S}$  for the graph  $(S, \mathcal{E})$  (a subgraph of the complete graph on vertex set  $S$ ). We refer to the triple  $(V, T, \mathbf{S})$  as a *boundary condition*.

In terms of generalising the proof of Theorem 4.1 to the RCM, the set  $V$  (respectively  $T$ ) represents the set of coloured vertices in  $B_n \setminus C_{29}$  that are connected by a coloured path to  $B_{0.5}$  (respectively, to  $\partial B_n$ ), before the vertices inside  $C_{29}$  have been added. The set  $S$  represents the remaining coloured vertices  $B_n \setminus C_{29}$ , and  $\mathcal{E}$  represents the set of edges between these vertices. However, this description is only for motivation, and the present section is self-contained; in particular, no colouring of vertices takes place in this section.

For  $\mu > 0$  and  $0 \leq r < s$ , let  $\mathcal{P}_{\mu,r,s}$  denote a homogeneous Poisson process of intensity  $\mu$  in  $A_{r,s}$ . Given  $(V, T, \mathbf{S})$  as described above, for  $0 \leq r < 29$  the RCM on  $\mathcal{P}_{\mu,r,29}$  with boundary condition  $(V, T, \mathbf{S})$  is obtained as follows:

we connect each pair of vertices  $x, y$  with  $x, y \in \mathcal{P}_{\mu, r, 29}$  or  $x \in \mathcal{P}_{\mu, r, 29}$  and  $y \in V \cup T \cup S$ , by an undirected edge with probability  $f(|x - y|)$ , independently of other pairs. For  $x \in \mathcal{P}_{\mu, r, 29}$  we then say  $x$  is *path-connected* to  $T$  (respectively, to  $V$ ) if there is a path from  $x$  to  $T$  (respectively,  $V$ ) using the edges created. If also  $y \in \mathcal{P}_{\mu, r, 29}$  then we say  $x$  is path-connected to  $y$  if there is a path from  $x$  to  $y$ , using the edges created along with the edges of  $\mathcal{E}$ .

Let  $V_r$ , respectively  $T_r$  be the set of vertices of  $\mathcal{P}_{\mu, r, 29}$  that are path-connected to  $V$ , respectively  $T$ . Let  $S_r$  be the set  $\mathcal{P}_{\mu, r, 29} \setminus (V_r \cup T_r)$ . Define the event

$$H(V, T, \mathbf{S}) := \{V_{20} \cap C_{21} \neq \emptyset\} \cap \{T_{20} \cap C_{21} \neq \emptyset\} \cap \{V_{20} \cap T_{20} = \emptyset\}. \quad (6.2)$$

Let  $H'(V, T, \mathbf{S})$  be the intersection of  $H(V, T, \mathbf{S})$  with the event that there exists  $v^* \in C_{20.1}$  and  $t^* \in C_{20.1}$  such that  $|v^* - t^*| > 1.5$  and  $V_{20} \cap C_{20.5} = \{v^*\}$  and  $T_{20} \cap C_{20.5} = \{t^*\}$ , and  $S_{20} \cap C_{20.5} = \emptyset$ . We can now state the main result of this section.

**Lemma 6.1** *Suppose  $f$  is non-increasing and (6.1) holds. Then there exists a continuous function  $\varepsilon : (0, \infty) \rightarrow (0, \infty)$  such that for any  $\mu \in (0, \infty)$ , and any boundary condition  $(V, T, \mathbf{S})$  we have*

$$P[H'(V, T, \mathbf{S})] \geq \varepsilon(\mu) P[H(V, T, \mathbf{S})]. \quad (6.3)$$

We shall need several further lemmas to prove Lemma 6.1. In these arguments, we often need build up the Poisson process  $\mathcal{P}_\mu$  in certain regions via a “scanning process”, as described in Meester, Penrose and Sarkar (1997) which gives a rigorous proof that it does indeed build up the Poisson process. For any set of vertices  $U$  and any point  $z \in \mathbb{R}^2$  let  $p(z, U)$  be the probability that a vertex at  $z$  is joined to at least one of the vertices in  $U$ . So  $1 - p(z, U) = \prod_{u \in U} (1 - f(|z - u|))$ .

We shall consider the process  $\mathcal{P}_{\mu, 24, 25}$  as the union of two independent half-intensity processes  $\mathcal{P}_{\mu/2, 24, 25}$  and  $\mathcal{P}'_{\mu/2, 24, 25}$ . Let  $E_1$  be the event that  $\mathcal{P}_{\mu/2, 24, 25}$  has precisely two elements, and one of these is connected to  $V_{25}$  while the other is connected to  $T_{25}$ , and  $V$  is not path-connected to  $T$  through  $\mathcal{P}_{\mu, 25, 29} \cup \mathcal{P}_{\mu/2, 24, 25} \cup S$ .

**Lemma 6.2** *For all boundary conditions  $(V, T, \mathbf{S})$ , it is the case that  $P[E_1] \geq 0.25 \exp(-25\pi\mu) P[H(V, T, \mathbf{S})]$ .*



**Proof.** Create the process  $\mathcal{P}_{\mu,25,29}$  and define the sets  $V_{25}$ ,  $T_{25}$  and  $S_{25}$  as described earlier. Then build up an inhomogenous process in from the edge of  $C_{25}$  (i.e. starting at distance 25 from  $x$  and working radially symmetrically inwards) with intensity  $\mu h_1(\cdot)$  where  $h_1(v) = p(v, V_{25})(1 - p(v, T_{25}))$ , until a vertex  $y$  occurs. Then add edges from  $y$  to  $V_{25}$  conditional on there being at least one such edge. Add edges independently from  $y$  to vertices in  $S_{25}$  in the usual way. Do not add any edges from  $y$  to  $T_{25}$ .

Now build up another inhomogenous process in from the edge of  $C_{25}$  with intensity  $\mu h_2(\cdot)$ , where  $h_2(v) = p(v, T_{25})(1 - p(v, V_{25}))$ , until a vertex  $z$  occurs. Add edges from  $z$  conditional on there being at least one edge from  $z$  to  $T_{25}$  and no edge from  $z$  to  $V_{25}$ .

Let  $E'_1$  be the event that we get such vertices  $y$  and  $z$  and  $y$  is not connected to  $z$  through  $S_{25}$ . Then  $E'_1$  must occur for the event  $H(V, T, \mathbf{S})$  to occur.

Let  $E''_1$  be the event that  $E'_1$  occurs with both  $y$  and  $z$  coming from the first half intensity process  $\mathcal{P}_{\mu/2,24,25}$  (rather than from  $\mathcal{P}'_{\mu/2,24,25}$ ). Then  $P[E''_1|E'_1] = 0.25$ . Given  $E''_1$  occurs, for  $E_1$  to occur we need only that there be no further vertices of  $\mathcal{P}_{\mu/2,24,25}$  besides  $y$  and  $z$ , and the conditional probability of this is at least  $\exp(-49\pi\mu/2)$ . Combining these probability estimates gives the result.  $\square$

Let  $\rho := \inf\{a > 0 : f(a) < 1\}$ , i.e. the radius of certain connection (this could be zero). We shall prove Lemma 6.1 separately for the two cases  $\rho < \frac{1}{\sqrt{2}} - 0.01$  and  $\rho \geq \frac{1}{\sqrt{2}} - 0.01$  (see Lemmas 6.4 and 6.6 below).

Suppose for now that  $\rho < \frac{1}{\sqrt{2}} - 0.01$ . Given  $y, z \in A_{24,25}$  with  $x, y, z$  not collinear, let  $b(y, z)$  be the point at distance 0.999 from  $y$ , at distance  $\rho + 0.01$  from  $xy$  and on the opposite side of the line  $xy$  to  $z$  (see Figure 6.1). Let  $b(z, y)$  be defined similarly. Define the region

$$Q(y, z) := C_{1.0001}(b(y, z)) \setminus (C_{25} \cup C_\rho(y))$$

and define  $Q(z, y)$  similarly (see Figure 6.1, where  $Q(z, y)$  is empty). The regions  $Q(y, z)$  and  $Q(z, y)$ , if non-empty, each have diameter less than 0.9 due to  $\rho$  being less than  $\frac{1}{\sqrt{2}} - 0.01$ .

Given  $y$  and  $z$  define  $T_{25}^{y,z}$  and  $V_{25}^{y,z}$  in the same manner as  $T_{25}$  and  $V_{25}$ , respectively, but using the point process  $\mathcal{P}_{\mu,25,29} \cup \{y, z\}$  instead of  $\mathcal{P}_{\mu,25,29}$ .

Suppose  $E_1$  occurs, and let  $y, z$  be the vertices of  $\mathcal{P}_{\mu/2,24,25}$ , with  $y$  path-connected to  $V$  and  $z$  path-connected to  $T$ . Let  $E_2$  be the event that there

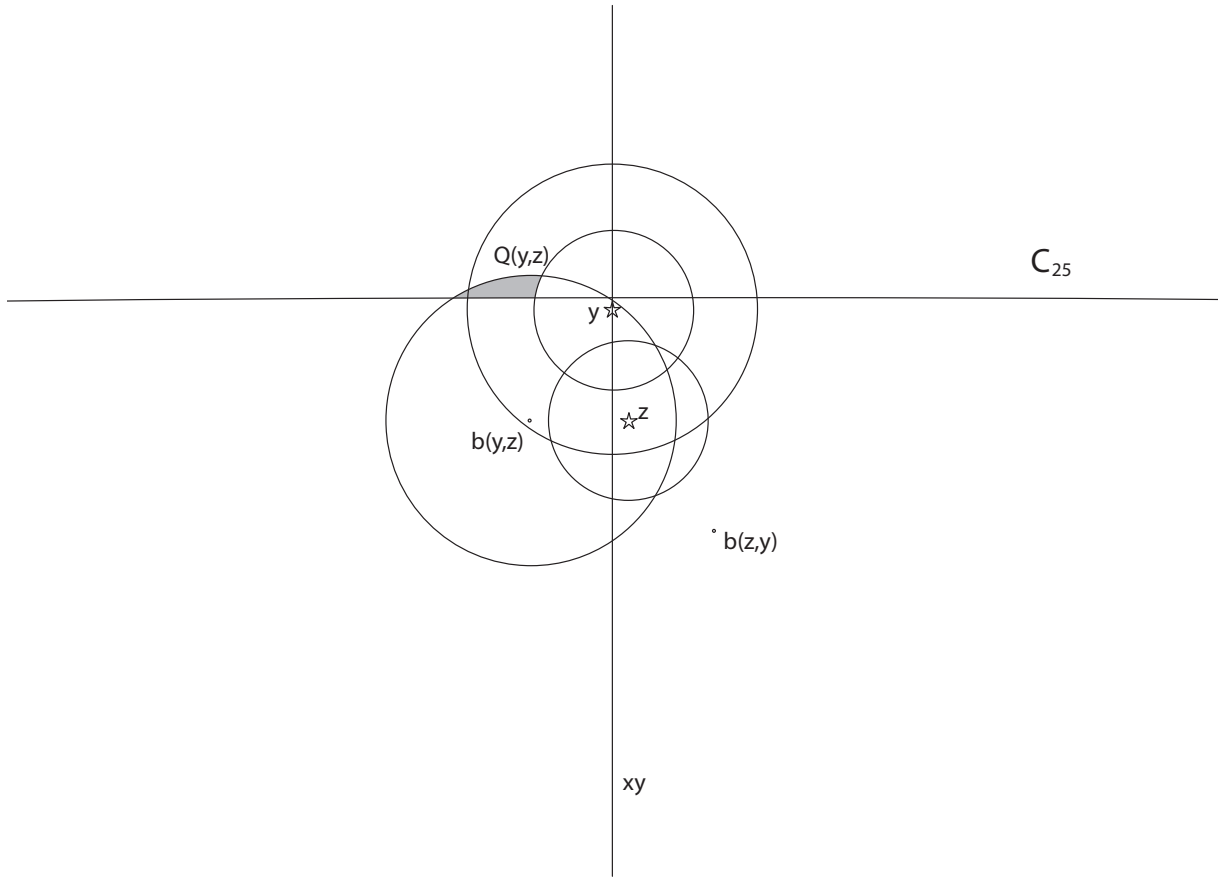


Figure 6.1: Here is a diagram showing the region  $Q(y, z)$  (in this case  $Q(z, y)$  is empty). The smaller circles are of radius  $\rho$  and the larger ones are of radius 1.0001

are no more than two vertices of  $T_{25}^{y,z}$  in  $Q(y, z)$  and no more than two vertices of  $V_{25}^{y,z}$  in  $Q(z, y)$ , and no vertices of  $\mathcal{P}_{\mu,25,29}$  at all, other than those of  $T_{25}^{y,z}$  and  $V_{25}^{y,z}$ .

**Lemma 6.3** *Suppose  $\rho < \frac{1}{\sqrt{2}} - 0.05$ . Then*

$$P[E_2|E_1] \geq f(0.9)^2 \exp(-29^2\pi\mu) =: \varepsilon_1(\mu).$$

**Proof.** The idea here is to condition on what happens inside the annulus  $A_{24,25}$ . The probability  $P[E_1]$  is the product of the probability that there are exactly two vertices in  $\mathcal{P}_{\mu/2,24,25}$ , and the probability that for two uniformly distributed vertices in  $A_{24,25}$ , they are joined one of them to  $T_{25}$  but not  $V_{25}$  and the other to  $V_{25}$  but not  $T_{25}$ . Given  $y$  and  $z$  in  $A_{24,25}$ , let  $I_{y,z}$  be the event (defined in terms of the Poisson process  $\mathcal{P}_{\mu,25,29}$  and associated edges) that  $y \in V_{25}^{y,z} \setminus T_{25}^{y,z}$  and  $z \in T_{25}^{y,z} \setminus V_{25}^{y,z}$ , and let  $p(y, z) = P[I_{y,z}]$  (this also depends on  $V, T$  and  $\mathbf{S}$ ). Then

$$P[E_1] = \exp(-49\pi\mu/2)(\mu/2)^2 \int_{A_{24,25}} \int_{A_{24,25}} p(y, z) dy dz.$$

Similarly,

$$P[E_1 \cap E_2] = \exp(-49\pi\mu/2)(\mu/2)^2 \int_{A_{24,25}} \int_{A_{24,25}} p'(y, z) dy dz,$$

where  $p'(y, z) = P[I'_{y,z}]$  and  $I'_{y,z}$  is the event that  $I_{y,z}$  occurs and also there are at most two vertices of  $T_{25}^{y,z}$  in  $Q(y, z)$ , and at most two vertices of  $V_{25}^{y,z}$  in  $Q(z, y)$ , and all vertices in  $\mathcal{P}_{\mu,25,29}$  are in  $V_{25,29}^{y,z} \cup T_{25,29}^{y,z}$ . Therefore we just need to show that  $p'(y, z) \geq \varepsilon_1 p(y, z)$  for Lebesgue-almost all  $y, z$  in  $A_{24,25}$ , and for all possible configurations where  $E_1$  occurs. We do this in stages.

**Stage 1.** Fix  $y$  and  $z$ . Let  $V_0 = V \cup \{y\}$  and  $T_0 = T \cup \{z\}$ . We now *exhaustively create* the set of vertices in  $A_{25,29} \setminus Q(z, y)$  that are path-connected to  $V_0$  but not to  $T_0$ , by which we mean the following sequence of steps. First create a process of intensity  $\mu p(\cdot, V_0)(1 - p(\cdot, T_0))$  in  $A_{25,29} \setminus Q(z, y)$ . Add edges from the new vertices to  $V_0 \cup S$  conditional on having at least one edge from each new vertex to  $V_0$  and no edges from the new vertices to  $T_0$ . Let  $V_1$  be the set of vertices outside  $V_0$  that are now path-connected to  $V_0$  (i.e. the newly added vertices and any vertices of  $S$  that are path-connected to

them). Next, create a process of intensity  $\mu p(\cdot, V_1)(1-p(\cdot, V_0))(1-p(\cdot, T_0))$  in  $A_{25,29} \setminus Q(z, y)$ , and add edges to these points conditional on having at least one edge from each new point to  $V_1$  but no edge to  $V_0$  or  $T_0$ . Let  $V_2$  be the set of points now path-connected to  $V_0$  that were not in  $V_0 \cup V_1$ . Next create a process in  $A_{25,29} \setminus Q(z, y)$  of intensity  $\mu p(\cdot, V_2)(1-p(\cdot, V_0 \cup V_1))(1-p(\cdot, T_0))$ .

Continue in this way, at each stage adding those vertices in  $A_{25,29} \setminus Q(z, y)$  that are connected to the latest  $V_i$  but not to earlier sets  $V_{i-1}, \dots, V_0$  or to  $T_0$ . At some stage this procedure must terminate (i.e. the new Poisson process has no points). This completes the exhaustive creation of points that are path-connected to  $V_0$  but not  $T_0$ .

Now let  $V'$  be the union of  $V$  with all vertices path-connected to  $V$  at this stage, and let  $U_y$  be the set of vertices path-connected to  $y$  at this stage.

**Stage 2.** Next, we exhaustively create (in a similar manner to the above) the set of vertices in  $A_{25,29} \setminus Q(y, z)$  that are path-connected to  $T_0$  but not to  $V' \cup U_y$ .

Now let  $T'$  be the union of  $T$  with all vertices path-connected to  $T$  at this stage, and let  $U_z$  be the set of all vertices path-connected to  $z$  at this stage.

**Stage 3.** Suppose next that  $z \notin T'$ . Otherwise, go on to Stage 4 below. Then, since we have exhaustively created the vertices connected to  $T' \cup U_z$  outside  $Q(y, z)$ , for  $I_{y,z}$  to occur there must be a vertex in  $Q(y, z)$  connected to  $T'$  and a vertex (possibly the same one) in  $Q(y, z)$  connected to  $U_z$ . Build up the process in  $Q(y, z)$  towards  $x$  with intensity

$$\mu p(\cdot, U_z) p(\cdot, T') [1 - p(\cdot, V' \cup U_y)]$$

until we get a vertex  $u$  (if any). If such a vertex occurs then we add edges from  $u$  to  $U_z$  and to  $T'$  conditional on there being at least one of each type, and add no edges from  $u$  to  $V' \cup U_y$ . We then let  $T''$  be the union of  $T'$  with all vertices now path-connected to  $T'$  (which will now include  $z$ ), and go to Stage 4 below.

If  $u$  does not occur, build up two more processes in  $Q(y, z)$ , with intensities

$$\mu [1 - p(\cdot, U_z)] p(\cdot, T') [1 - p(\cdot, V' \cup U_y)]$$

and

$$\mu p(\cdot, U_z) [1 - p(\cdot, T')] [1 - p(\cdot, V' \cup U_y)]$$

until we get vertices  $u_1$  and  $u_2$  respectively. If we get such vertices then  $u_1$  will be joined to  $T'$  and  $u_2$  will be joined to  $U_z$ . Also,  $u_1$  will be joined to

$u_2$  with probability at least  $f(0.9)$ . Assume this happens (so now  $z$  is path-connected to  $T$ ), and let  $T'' := T' \cup U_z \cup \{u_1, u_2\}$  and go to Stage 4. If we do not get  $u_1$  and  $u_2$ , then  $I_{y,z}$  cannot occur.

**Stage 4.** Suppose now that  $y \notin V'$ . Otherwise, go on to Stage 5 below. Build up the process in  $Q(z, y)$  towards  $x$  with intensity

$$\mu p(\cdot, U_y) p(\cdot, V') [1 - p(\cdot, T'' \cup U_z)]$$

until we get a vertex  $w$ . If such a vertex occurs, then add edges from  $w$  to  $U_y$  and to  $V'$  conditional on there being at least one of each type, add none to  $T' \cup U_z$ . We now have a path from  $y$  to  $V$  and go to Stage 5 below.

If  $w$  does not occur, build up two more processes in  $Q(z, y)$ , with intensities

$$\mu [1 - p(v, U_y)] p(v, V') [1 - p(v, T'' \cup U_z)]$$

and

$$\mu p(v, U_y) [1 - p(v, V')] [1 - p(v, T'' \cup U_z)]$$

until we get vertices  $w_1$  and  $w_2$  respectively. If we get such vertices, then  $w_1$  will be joined to  $V'$  and  $w_2$  will be joined to  $U_y$ . Also  $w_1$  will be joined to  $w_2$  with probability at least  $f(0.9)$ . Assume this happens (so then we have a path from  $y$  to  $V$ ), and go to Stage 5. If  $w_1$  and  $w_2$  do not occur, then  $I_{y,z}$  cannot occur.

**Stage 5.** By now we have  $y$  connected (by a path) to  $V$  and  $z$  connected to  $T$ , and  $V$  not connected to  $T$ . Now sample the rest of  $\mathcal{P}_{\mu, 25, 29}$ . Then as long as no more vertices occur when we do this (an event with probability at least  $\exp(-29^2 \pi \mu)$ ), event  $I'_{y,z}$  occurs. Therefore, we have shown that  $p'(y, z) \geq \varepsilon_1 p(y, z)$ , as required.  $\square$

**Lemma 6.4** *Suppose that  $f$  is non-increasing and (6.1) holds, and that  $\rho < \frac{1}{\sqrt{2}} - 0.05$ . Then the conclusion of Lemma 6.1 holds.*

**Proof.** Suppose  $E_1 \cap E_2$  occurs, and let  $y$  and  $z$  be as in the definition of  $E_1$  (i.e. the points in  $\mathcal{P}_{\mu/2, 24, 25}$  that are path-connected to  $V$  and to  $T$  respectively). Let  $b_1 = b(y, z)$  and  $a_1 = b(z, y)$ . Define discs  $D_1 := C_{0.0001}(b_1)$  and  $K_1 := C_{0.0001}(a_1)$ . Then

$$\begin{aligned} \min(\text{dist}(D_1, z), \text{dist}(K_1, y), \text{dist}(D_1, K_1)) &\geq \rho + 0.005; \\ \min(\text{dist}(D_1, \mathbb{R}^2 \setminus C_{25}), \text{dist}(K_1, \mathbb{R}^2 \setminus C_{25})) &\geq \max(\rho + 0.005, 0.6), \end{aligned} \quad (6.4)$$

and for any  $b' \in D_1$  and  $a' \in K_1$  we have  $\max(|b' - y|, |a' - z|) \leq 0.9991$ .

Next take further discs  $D_i = C_{0.0001}(b_i)$  and  $K_i = C_{0.0001}(a_i)$ , for  $2 \leq i \leq 7$ , such that each of these discs is contained in  $A_{20,24}$ , and discs  $D_1, K_1, \dots, D_7, K_7$  are disjoint, and

$$\begin{aligned} |b_i - b_{i-1}| &= |a_i - a_{i-1}| = 0.999, \quad 2 \leq i \leq 7; \\ \min(\text{dist}(D_2, K_1), \text{dist}(K_2, D_1)) &\geq \rho + 0.005; \\ \text{dist}(D_i, K_j) &\geq 1.1, \quad 1 \leq i, j \leq 7, (i, j) \notin \{(1, 1), (1, 2), (2, 1)\}; \\ \min(|b_i - x|, |a_i - x|) &\geq 20.6, \quad 2 \leq i \leq 6; \end{aligned}$$

and  $|b_7 - x| = |a_7 - x| = 20.05$  and  $|b_7 - a_7| \geq 1.5$ .

Now create the Poisson process  $\mathcal{P}'_{\mu/2, 24, 25} \cup \mathcal{P}_{\mu, 20, 24}$ . Let  $E_3$  be the event that we get exactly one new vertex in each of  $D_i$  and  $K_i$  (denoted  $y_i$  and  $z_i$  respectively) for  $1 \leq i \leq 7$ , and no other new vertices. Then

$$P[E_3 | E_1 \cap E_2] \geq ((0.0001)^2 \pi \mu / 2)^{14} \exp(-25^2 \pi \mu) =: \varepsilon_2. \quad (6.5)$$

Now, assuming  $E_1 \cap E_2 \cap E_3$  occurs, decide which edges occur involving the new vertices. The probability that we get edges forming the paths  $(y, y_1, y_2, \dots, y_7)$  and  $(z, z_1, \dots, z_7)$  is at least  $f(0.9991)^{14}$ .

By (6.4), the probability that  $y_1$  is not joined to  $z$ ,  $z_1$  or  $z_2$  is at least  $[1 - f(\rho + 0.005)]^3$ . Also for the probability that  $y_1$  is joined to no vertices of  $T_{25}^{y,z} \cap A_{25,29}$  is at least  $[1 - f(\rho + 0.005)]^2$ , because at most 2 such vertices lie in  $Q(y, z)$  since event  $E_2$  is assumed to occur, and no such vertices lie within  $\rho$  of  $y$  since event  $E_1$  is assumed to occur, and all other such vertices are more than unit distance from  $y_1$ .

Similarly,  $z_1$  is not connected to  $y$  or  $y_2$  or any vertex in  $T_{25}^{y,z} \cap A_{25,29}$  with probability at least  $[1 - f(\rho + 0.005)]^4$  given  $E_1 \cap E_2$ .

If  $y_2$  is not connected to  $z_1$  and  $z_2$  is not connected to  $y_1$ , then for  $2 \leq i \leq 7$ , none of the vertices  $y_i$  can be connected to any of the vertices  $z_j$  or to  $T_{25}^{y,z} \cap A_{25,29}$ , and none of the vertices  $z_i$  can be connected to any of the vertices  $y_j$  or to  $V_{25}^{y,z} \cap A_{25,29}$ . Therefore, we arrive at

$$P[H'(T, V, \mathbf{S}) | E_1 \cap E_2 \cap E_3] \geq f(0.9991)^{14} [1 - f(\rho + 0.005)]^9 := \varepsilon_3.$$

Hence, by (6.5) and Lemmas 6.2 and 6.3, taking  $\varepsilon = 0.25 \exp(-25\pi\lambda) \varepsilon_1 \varepsilon_2 \varepsilon_3$ , we have the desired result (6.3) for  $\rho \leq \frac{1}{\sqrt{2}} - 0.05$ .  $\square$

Now, to complete the proof of Lemma 6.1 we look at the case where  $\rho \geq \frac{1}{\sqrt{2}} - 0.05$ . We create the process  $\mathcal{P}_{\mu,25,29}$  and define  $V_{25}, T_{25}$  and  $S_{25}$  as before. Let  $E_4$  be the event that  $V_{25}$  and  $T_{25}$  are disjoint. This must occur for  $E_1$  to occur.

We then add the half intensity process  $\mathcal{P}_{\mu/2,24,25}$ . Let  $F_V$  be the event that  $E_4$  occurs and there is just one vertex  $y$  of  $\mathcal{P}_{\mu/2,24,25}$ , and it is connected to  $V_{25}$  but not  $T_{25}$ , and  $T_{25}$  includes a vertex in  $A_{|y-x|,|y-x|+.05}$ . Similarly, let  $F_T$  be the event that  $E_4$  occurs and there is just one vertex  $y$  of  $\mathcal{P}_{\mu/2,24,25}$ , and it is connected to  $T_{25}$  but not  $V_{25}$ , and  $V_{25}$  includes a vertex in  $A_{|y-x|,|y-x|+.05}$ .

Let  $G_V$  be the event that  $E_4$  occurs and there are just two vertices  $y, z$  of  $\mathcal{P}_{\mu/2,24,25}$ , and  $y$  is connected to  $V_{25}$  but not  $T_{25}$  and  $z$  is connected to  $T_{25}$  but not  $V_{25}$ , and  $|y-x| > |z-x|$  and  $T_{25} \cap A_{|y-x|,|y-x|+.05} = \emptyset$ . Similarly let  $G_T$  be the event that  $E_4$  occurs and there are two vertices  $y, z$  of  $\mathcal{P}_{\mu/2,24,25}$ , and  $y$  is connected to  $T_{25}$  but not  $V_{25}$  and  $z$  is connected to  $V_{25}$  but not  $T_{25}$ , and  $|y-x| > |z-x|$  and  $V_{25} \cap A_{|y-x|,|y-x|+.05} = \emptyset$ .

**Lemma 6.5** *Let  $\varepsilon_4(\mu) := 0.25 \exp(-25\pi\mu)$ . Then for any boundary conditions  $(V, T, \mathbf{S})$  we have*

$$P[H(V, T, \mathbf{S})] \leq \varepsilon_4^{-1}(P[F_V] + P[F_T] + P[G_V] + P[G_T]). \quad (6.6)$$

**Proof.** After creating the process  $\mathcal{P}_{\mu,25,29}$ , we build a process of intensity

$$\mu(p(\cdot, V_{25})(1 - p(\cdot, T_{25})) + p(\cdot, T_{25})(1 - p(\cdot, V_{25})))$$

inwards into  $C_{25}$ , until we get a vertex  $y \in A_{24,25}$ . Let  $E'$  be the event that such a vertex occurs. Event  $E'$  must occur for  $H(V, T, \mathbf{S})$  to occur.

If  $E'$  occurs, add edges from  $y$  to  $V_{25} \cup T_{25} \cup S_{25}$ , conditional on there being at least one edge from  $y$  to  $V_{25} \cup T_{25}$  but there not being edges from  $y$  both to  $T_{25}$  and to  $V_{25}$ .

Suppose for now that  $y$  is connected to  $V_{25}$  (we call this event  $E'_V$ ). Let  $F'_V$  be the event that there is a vertex of  $T_{25}$  in the thin annulus  $A_{|y-x|,|y-x|+.05}$ . If  $F'_V$  occurs, then if  $y$  comes from in the first half-intensity process  $\mathcal{P}_{\mu/2,24,25}$  and there are no further vertices from  $\mathcal{P}_{\mu/2,24,25}$  (an event of probability at least  $\varepsilon_4$ ), event  $F_V$  occurs.

If  $F'_V$  does not occur, then let  $V_{25}^y$  denote the set of points of  $\mathcal{P}_{\mu,25,29} \cup \{y\}$  that are path-connected to  $V$ , and build a process of intensity  $\mu p(\cdot, T_{25})(1 - p(\cdot, V_{25}^y))$ , inwards inside  $C_{|y-x|}$ , until we get a vertex  $z \in A_{24,|y-x|}$  (this must happen if  $E'_V \cap H(V, T, \mathbf{S})$  is to occur but  $F'_V$  does not occur). If then  $y$  and

$z$  both come from  $\mathcal{P}_{\mu/2,24,25}$  and there are no further vertices in  $\mathcal{P}_{\mu/2,24,25}$  (an event of probability at least  $\varepsilon_4$ ), then  $G_V$  occurs. Combining these yields

$$\begin{aligned} P[F_V] + P[G_V] &\geq \varepsilon_4(P[E'_V \cap F'_V] + P[E'_V \cap H(V, T, \mathbf{S}) \setminus F'_V]) \\ &\geq \varepsilon_4 P[E'_V \cap H(V, T, \mathbf{S})]. \end{aligned}$$

If  $E' \setminus E'_V$  occurs, then  $y$  is connected to  $T_{25}$  and a similar argument yields

$$P[F_T] + P[G_T] \geq \varepsilon_4 P[(E' \setminus E'_V) \cap H(V, T, \mathbf{S})],$$

and combining the last two estimates gives us (6.6).  $\square$

The following result, combined with Lemma 6.4, completes the proof of Lemma 6.1.

**Lemma 6.6** *Suppose that  $f$  is non-increasing and (6.1) holds, and that  $\rho \geq \frac{1}{\sqrt{2}} - 0.05$ . Then the conclusion of Lemma 6.1 holds.*

**Proof.** If  $F_V$  or  $F_T$  occurs we can continue in similar fashion to the argument for Gilbert's graph. Suppose  $F_V$  occurs, let  $y$  be as in the definition of  $F_V$  and set  $r = |y - x|$ , and let  $z$  be an arbitrarily chosen vertex of  $T_{25}$  lying in  $A_{r, r+0.05}$ .

Let  $a$  be the point with  $|a - y| = \rho$  and  $|a - x| = r$ , on the other side of  $y$  to  $z$  (see Figure 6.2). Let  $b$  be the point with  $|b - z| = \rho$  and  $|b - x| = r$ , lying on the other side of  $z$  to  $y$ . Let  $a_1$  be the point in  $C_r$  with  $|a_1 - y| = 0.99$  and  $|a_1 - a| = 1.01$  and let  $D_1 := C_{0.001}(a_1)$ . Similarly let  $b_1$  be the point in  $C_r$  with  $|b_1 - z| = 0.99$  and  $|b_1 - b| = 1.01$ , and let  $K_1 := C_{0.001}(b_1)$ . Note that  $|y - z| > \rho$  so  $\text{dist}(D_1, K_1) > \rho + 0.01$ .

Let  $D_2, \dots, D_7$  and  $K_2, \dots, K_7$  be discs of radius 0.001 and successive centres distant 0.99 from each other, such that, as before, having exactly one red vertex in each of these little circles and no other vertices in  $A_{20,25}$ , and connections between the vertices in successive circles  $D_i, D_{i+1}$  and  $K_j, K_{j+1}$  ensures that  $H'(V, T, \mathbf{S})$  occurs.

Now sample  $\mathcal{P}'_{\mu/2,24,25} \cup \mathcal{P}_{\mu,20,24}$  and consider the event  $E_5$ , that there is exactly one new vertex  $y_i$  in  $D_i$  and exactly one new vertex  $z_i$  in  $K_i$  for  $1 \leq i \leq 7$ , and no other new vertices. Then

$$P[E_5 | F_V] \geq (0.001^2 \pi \mu)^{14} \exp(-25^2 \pi \mu) =: \varepsilon_5.$$

Next, decide which edges are created from the new vertices. We want  $y_1$  to connect with  $y$  (which happens with probability at least  $f(0.991)$ ) but



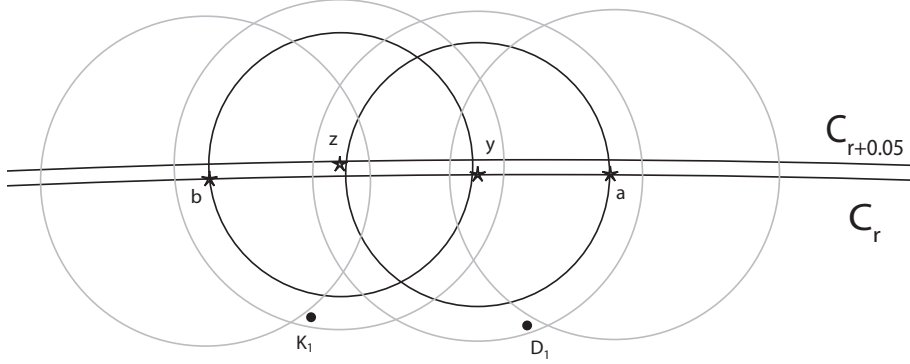


Figure 6.2: The grey circles are of radius 1 and the black circles are of radius  $\rho$ .

not to any vertices in  $T_{25}^{y,z}$  (which cannot happen as  $a$  is the closest place for a vertex in  $T_{25}^{y,z}$ ). Similarly we also want  $z_1$  to connect with  $z$  but not to  $V_{25}^{y,z}$ . Also we want  $z_1, y_1$  to not be joined, which happens with probability at least  $1 - f(\rho + 0.01)$ . We also want connections between vertices in successive circles  $D_i, D_{i+1}$  and  $K_i, K_{i+1}$ . Given  $F_V \cap E_5$ , these events all happen with probability at least  $f(0.991)^{14}(1 - f(\rho + 0.01))$ , in which case  $H'(V, T, S)$  occurs; hence

$$P[H'(V, T, \mathbf{S})|F_V] \geq f(0.991)^{14}[1 - f(\rho + 0.01)]\varepsilon_5 := \varepsilon_6 \quad (6.7)$$

and similarly  $P[H'(V, T, \mathbf{S})|F_T] \geq \varepsilon_6$ .

Now suppose the event  $G_V$  occurs. Then with  $r := |y - x|$ , we have  $z$  inside  $C_r$  connected to a vertex  $z_0$  of  $T_{25}$  which must be outside  $C_{r+0.05}$ .

Let  $a$  be the point with  $|a - y| = \rho$  and  $|a - x| = r + 0.05$  on the opposite side of  $y$  to  $z_0$ . Let  $l_a$  be the arc of  $C_{r+0.05}$  to the left of  $a$  (see Figure 6.3).

Let  $b'(y, z)$  be the point at distance 0.999 from  $y$  and  $\rho + 0.01$  from  $l_a$ , and define the region

$$Q'(y, z) := C_{1.0001}(b'(y, z)) \setminus (C_\rho(y) \cup C_{r+0.05}(x)).$$

The diameter of  $Q'(y, z)$  is less than 0.9, due to  $\rho$  being at least  $2^{-1/2} - 0.05$ . Let  $E_6$  be the event that there are no more than 2 vertices of  $T_{25}^{y,z}$  in  $Q'(y, z)$ ,

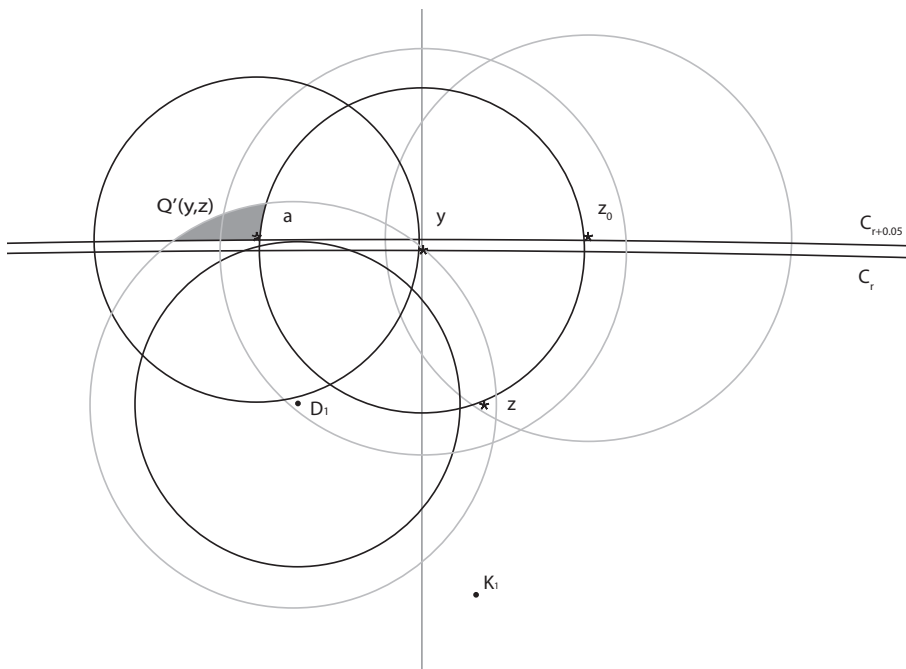


Figure 6.3: The grey circles are of radius 1 and the black circles are of radius  $\rho$ .

and no other vertices than those of  $T_{24}^{y,z} \cup V_{24}^{y,z}$  in  $A_{25,29}$ . By a similar argument to the proof of Lemma 6.3 the conditional probability of  $E_6$  satisfies

$$P[E_6|G_V] \geq f(0.9) \exp(-29^2\mu) =: \varepsilon_7. \quad (6.8)$$

Set  $D_1 = C_{0.0001}(b'(y, z))$ . If there is a vertex in  $D_1$  it will be distant at least  $\rho + 0.001$  from  $z$  and from any vertex in  $T_{25}^{y,z}$  (as  $l_a$  is the closest place such a vertex can be given  $G_V$  occurs) and at most 0.9991 from  $y$ . Let  $a_1$  be the point distant 0.999 from  $z$  on the line parallel with  $xy$  through  $z$ , and let  $K_1 := C_{0.0001}(a_1)$ . We can then pick little discs  $D_i, K_i$ ,  $2 \leq i \leq 7$ , of radius 0.0001, as before (see Figure 5.1) such that if there is exactly one vertex in each of these discs and no other vertices in the rest of  $\mathcal{P}_{20,25}$ , and connections between vertices in successive discs, then  $H'(V, T, S)$  occurs.

Suppose that for the Poisson process  $\mathcal{P}_{\mu,20,24} \cup \mathcal{P}'_{\mu/2,24,25}$  there is exactly one vertex  $y_i \in D_i$  and exactly one vertex  $z_i \in K_i$  for each  $i$  and there are no other vertices. This has probability at least  $[0.0001^2\pi\mu/2]^{14} \exp(-25^2\pi\mu)$ . Given this event, consider now the event that we get all connections occurring along the paths  $(y, y_1, \dots, y_7)$  and  $(z, z_1, \dots, z_7)$  but no connection from  $y_1$  to to any vertex in  $T_{25}^{y,z}$ . This has probability at least  $f(0.9991)^{14}[1 - f(\rho + 0.001)]^3$  (assuming  $E_6$  occurs), and if this occurs then  $H'(V, T, \mathbf{S})$  occurs. Hence,  $P[H'(V, T, \mathbf{S})|E_6 \cap G_V] \geq \varepsilon_8$ , with

$$\varepsilon_8 := [0.0001^2\pi\mu/2]^{14} \exp(-25^2\pi\mu) f(0.9991)^{14} [1 - f(\rho + 0.001)]^3 \leq \varepsilon_6,$$

and similarly  $P[H'(V, T, \mathbf{S})|E_6 \cap G_T] \geq \varepsilon_8$ .

Combined with (6.7) and (6.8), this gives us (for  $\rho \geq \frac{1}{\sqrt{2}} - 0.05$ ) the bound

$$4P[H'(V, T, \mathbf{S})] \geq \varepsilon_7\varepsilon_8(P[G_V] + P[G_T] + P[F_V] + P[F_T]).$$

Combined with Lemma 6.5, this gives us the desired result (6.3) with  $\varepsilon = 0.25\varepsilon_4\varepsilon_7\varepsilon_8$ .  $\square$

## 6.2 Proof of Theorem 4.2

We now generalise Theorem 4.1 to the Random Connection Model with connection function  $f : [0, \infty) \rightarrow [0, 1]$ , where  $f$  is non-increasing and has bounded support. Without loss of generality we assume (6.1) holds (as if not we can rescale). Fix  $p, q$  and  $\lambda > \lambda_f$ . For the enhancement this time we

say that a vertex  $x$  is correctly configured if it is closed and there are only 4 vertices  $v, w, y, z$  within 1 of  $x$ , they are all red and joined to  $x$  and  $v \sim w$  and  $y \sim z$  but there are no other edges amongst  $v, w, y, z$ . If a vertex is correctly configured it is made green independently with probability  $q$ . Notice that another vertex could be not joined to  $x$  but still cause it to be incorrectly configured by being within 1 of it. Again let  $A_n$  be the event that for the Poisson process  $\mathcal{P}_\lambda \cap B_n$ , there is a path from a coloured vertex in  $B_{0.5}$  to a coloured vertex in  $\partial B_n$  in  $RCM(\mathcal{P}_\lambda, f) \cap B_n$  using only coloured vertices (note that  $A_n$  is based on a process completely inside  $B_n$ ; we do not allow vertices outside of  $B_n$  to affect possible enhancements inside  $B_n$ ). Again let  $\theta_n(p, q)$  be the probability that  $A_n$  occurs, and define

$$\theta(p, q) \equiv \liminf_{n \rightarrow \infty} (\theta_n(p, q)).$$

The following proposition is the equivalent of Proposition 5.1 and the proof easily carries over to this model.

**Proposition 6.1** *If  $f$  is non-increasing and satisfies (6.1) then there is a.s. an infinite connected component in  $RCM(\mathcal{P}_\lambda, f)$  using only red and green vertices if and only if  $\theta(p, q) > 0$ .*

Again define  $x$  to be 1-pivotal in  $B_n$  if putting  $Y_0 = 0$  means that  $A_n^x$  occurs but putting  $Y_0 = 1$  means it does not and define  $x$  to be 2-pivotal in  $B_n$  if inserting a vertex at  $x$  and putting  $Z_0 = 0$  means  $A_n^x$  occurs but putting  $Z_0 = 1$  means it does not.

The following lemma is the equivalent of Lemma 5.1.

**Lemma 6.7** *For all  $n > 0.5$  and  $p \in (0, 1)$  and  $q \in (0, 1)$  it is the case that*

$$\frac{\partial \theta_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,1}(x, p, q) \, dx \quad (6.9)$$

and

$$\frac{\partial \theta_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,2}(x, p, q) \, dx. \quad (6.10)$$

The proof of Lemma 5.1 easily carries over to this more general case. Next is a proof of the equivalent of Lemma 5.2 for the Random Connection Model under the current assumptions.

**Lemma 6.8** *Suppose  $f$  is non-increasing and (6.1) holds. Then there is a continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $(p, q) \in (0, 1)^2$ ,  $n > 100$  and  $x \in B_n$ ,*

$$P_{n,2}(x) > \delta(p, q)P_{n,1}(x).$$

In the proof we again write  $C_r$  for  $C_r(x)$ . Also we define events  $E_{n,1}(x)$  and  $R_n(x, \alpha, \beta)$  as in Section 5. It can easily be seen that the proof of Lemma 5.3 extends to this case as again the number of possible green vertices in the completed process in a bounded region is bounded. Therefore

$$P[E_{n,1}(x) \cap R_n(x, 20, 30)] \geq \delta_1 P_{n,1}(x). \quad (6.11)$$

Assume for now that  $30.5 < |x| < n - 30.5$ . Now suppose we create the whole process of intensity  $\lambda$  in  $B_n \setminus C_{30}$  and the red process of intensity  $p\lambda$  in the annulus  $A_{29,30}$ . We decide which vertices outside  $C_{30}$  are red, and assuming no other vertices occur in  $A_{29,30}$ , we then know which vertices outside  $C_{30}$  are correctly configured. We then determine which of these are green.

At this stage, let  $V$  be the set of coloured vertices in  $B_n \setminus C_{29}$  that are connected (by a coloured path) to  $B_{0.5}$  and let  $T$  be the coloured vertices in  $B_n \setminus C_{29}$  that are connected to  $\partial B_n$ . Let  $S$  be the remaining coloured vertices in  $B_n \setminus C_{29}$ , and let  $\mathcal{E}$  be the set of edges on  $S$  inherited from the original random connection model. Set  $\mathbf{S} := (S, \mathcal{E})$ .

Then we can apply Lemma 6.1, using these boundary conditions, to the Poisson process of red vertices, of intensity  $\mu = \lambda p$  inside  $C_{29}$ . If  $E_{n,1}(x) \cap R_n(x, 20, 30)$  occurs, then  $H(V, T, \mathbf{S})$  must occur, and therefore by Lemma 6.1,

$$P[H'(V, T, \mathbf{S})] \geq \varepsilon(\lambda p)\delta_1 P_{n,1}(x).$$

Now we can find  $\delta_2$  such that given  $H'(V, T, \mathbf{S})$  occurs, the probability of  $x$  being 2-pivotal is at least  $\delta_2$ . Indeed, with  $y^*$  and  $z^*$  as in the definition of  $H'(V, T, S)$ , we just find little discs  $D_1, \dots, D_{30}$  and  $K_1, \dots, K_{30}$  of radius 0.005 leading from  $y^*$  and  $z^*$  in towards a bow-tie configuration around  $x$  such that having one red vertex in each of these discs, with connections between successive discs, and no other vertices inside  $C_{20}$ , no vertices in the non-red process inside  $C_{30}$ , and having  $Y_0 > p$  ensures  $x$  is 2-pivotal. This all occurs with probability at least

$$\delta_2 := (0.005^2 \pi \lambda p)^{60} [f(0.9)]^{64} \exp(-900\pi\lambda)(1-p).$$

Therefore we have

$$P_{n,2}(x) \geq \delta_1 \delta_2 \varepsilon P_{n,1}(x)$$

for  $30.5 < |x| < n - 30.5$ .

If  $|x| \leq 30.5$  or  $|x| \geq n - 30.5$ , then by minor modifications of the last part of the proof of Lemma 5.2 we can find some continuous  $\delta_3 : (0, 1)^2 \rightarrow (0, \infty)$  such that  $P_{n,2}(x) \geq \delta_3(p, q)P_{n,1}(x)$ . So taking  $\delta = \delta_1\delta_2\delta_3\varepsilon$  will give us the result.  $\square$

The following proposition follows immediately from Lemmas 6.7 and 6.8.

**Proposition 6.2** *There is a strictly positive continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n \geq 100$  and  $(p, q) \in (0, 1)^2$ , we have*

$$\frac{\partial \theta_n(p, q)}{\partial q} \geq \delta(p, q) \frac{\partial \theta_n(p, q)}{\partial p}.$$

**Proof of Theorem 4.2.**

Set  $p^* = p_c^{\text{site}}$  and  $q^* = (1/8)(p^*)^2$ . Using Proposition 6.2 and looking at a small box around  $(p^*, q^*)$ , we can argue as in the last part of section 5 to find  $p < p^*$  such that the enhanced model with parameters  $(p, p^2)$  percolates, i.e. has an infinite coloured component, almost surely. Again this can be coupled with a bond percolation process with parameter  $p$  in such a way that if the enhanced model percolates almost surely then so does the bond process. Therefore  $p_c^{\text{bond}} \leq p < p_c^{\text{site}}$ .  $\square$

### 6.3 Proof of Theorems 4.4 and 4.5

As we are now using an underlying Poisson point process for the rest of the thesis we now write  $RCM(\lambda, f)$  instead of  $RCM(\mathcal{P}_\lambda, f)$  to denote the graph arising from the random connection model with connection function  $f$  and Poisson intensity  $\lambda$ . For proving Theorem 4.4, it is useful to consider mixed bond-site percolation on the graph  $RCM(\lambda, f)$ . Each site is open with probability  $p$ , and each bond is open with probability  $q$ . Clearly the graph resulting from performing this mixed percolation process on  $RCM(\lambda, f)$  may be viewed as a realisation of  $RCM(p\lambda, qf)$ .

In proving Theorem 4.4 we assume without loss of generality that (6.1) holds. We consider a new site percolation model, where sites are open with probability  $pq$  if they are correctly configured and with probability  $p$  if they are not correctly configured. Each site is designated either an *up-site* or a *down-site*, each with probability  $1/2$ , independently of everything else. We say that vertex  $y$  is a 1-neighbour of vertex  $x$  if  $|y - x| \leq 1$ . A vertex  $x$  is

correctly configured if it has exactly two 1-neighbours (denoted  $y_1$  and  $y_2$ , say) and  $x$  is connected both to  $y_1$  and to  $y_2$ , and  $x$  is a down-site but  $y_1$  and  $y_2$  are up-sites (see Figure 6.4).

The extra randomization of up-sites and down-sites is designed to ensure that if a site is correctly configured, then its neighbours are not.

We build this model by having a Poisson process of intensity  $\lambda$  and labelling vertices  $x_1, x_2, \dots$  in order of distance from the origin. We also have independent uniform random variables  $W_i, Y_i, Z_i$  for  $i = 0, 1, 2, \dots$ . We say that vertex  $x_i$  is an up-site if and only if  $W_i < 1/2$ . If a vertex  $x_i$  is correctly configured it is open if  $Y_i < p$  and  $Z_i < q$ . Otherwise it is open if  $Y_i < p$ . We define  $\partial B_n$  to be  $B_n \setminus B_{n-0.2}$ . We let  $A_n$  be the event that there is an open path from  $B_{0.2}$  to  $\partial B_n$  in the process restricted to  $B_n$ , and for  $x \in B_n$  define  $A_n^x$  similarly in terms of the process in  $B_n$  with an added vertex at  $x$ .

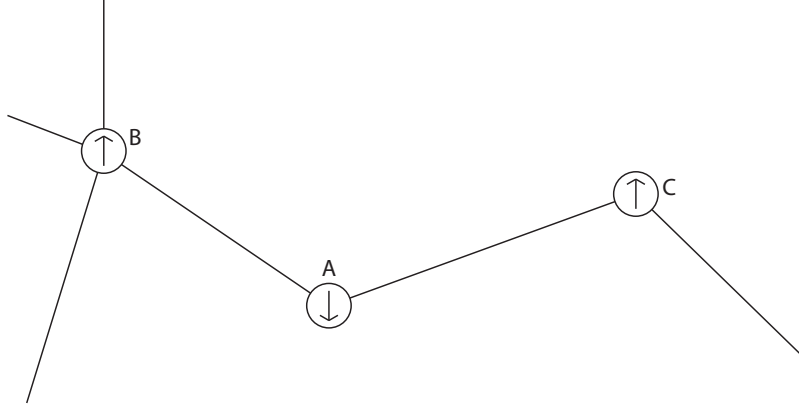


Figure 6.4: Here the vertex  $A$  is correctly configured but  $B$  and  $C$  are not.

Let the sites  $x_i$  for which  $Y_i < p$  be denoted *red* (a standard Bernoulli site percolation process). The set of open sites may be viewed as a *diminishment* of the set of red sites, in which each correctly configured red site is removed with probability  $1 - q$ . We can couple the diminished site percolation process to the mixed bond-site process (with parameters  $p, q$ ) in such a way that if the mixed process percolates then so does the diminished site process, as follows.

List the edges of this graph in an arbitrary order as  $e_1, e_2, \dots$ , and determine the open sites and edges for the mixed bond-site process. Deem each

vertex to be red if and only if it is open in the mixed process. If a vertex  $x$  is correctly configured, then it has degree 2 and has no correctly configured neighbour. In this case, let  $x$  be diminished (i.e. removed from the set of red vertices) if and only if the first edge incident to it (according to the given ordering) is closed.

If there is an infinite open path in the mixed percolation process, we can find such a path which starts at a non-correctly configured vertex. In this case, every vertex in the path will be red and undiminished, so there will be an infinite path in the diminished site percolation process as well.

Let  $\theta_n(p, q)$  be the probability that  $A_n$  occurs and let  $\theta(p, q)$  be the limit inferior. The proof of Proposition 5.1 is easily modified to this model. We say that a point  $x$  is 3-pivotal if putting a vertex at  $x$  and making  $Y_0 < p$  means that  $A_n^x$  occurs but having  $Y_0 > p$  means it does not. Similarly with 4-pivotal and  $Z_0$  and  $q$ . Again we have a form of the Margulis-Russo formulae:

$$\frac{\partial \theta_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,3}(x, p, q) dx \quad (6.12)$$

and

$$\frac{\partial \theta_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,4}(x, p, q) dx. \quad (6.13)$$

We then need to prove the equivalent of Lemma 5.2:

**Lemma 6.9** *There is a function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n > 100$  and all  $x \in B_n$  we have*

$$P_{n,4}(x, p, q) \geq \delta(p, q) P_{n,3}(x, p, q). \quad (6.14)$$

Before proving this, we give the equivalent of Lemma 5.3 which says we can assume all the vertices in an annulus of fixed size are red and none of them are diminished. Given  $p$  and  $q$ , and given  $1 < \alpha < \beta$ , let  $R_n(x, \alpha, \beta)$  be the event that all vertices in  $A_{\alpha, \beta}(x)$  are red. Let  $R'_n(x, \alpha, \beta)$  be the event that  $R_n(x, \alpha, \beta)$  occurs and also none of the vertices in  $A_{\alpha-1, \beta+1}(x)$  is diminished.

**Lemma 6.10** *There exists continuous  $\delta_1 : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n > 100$  and  $x \in B_n$  we have*

$$P[E_{n,3}(x) \cap R'_n(x, \alpha, \beta)] \geq \delta_1(p, q) P_{n,3}(x, p, q). \quad (6.15)$$



**Proof.** We create the whole Poisson point process of intensity  $\lambda$  in  $B_n$ , and the edges between these vertices, and decide which vertices outside  $A_{\alpha-1,\beta+1}$  are red, and which of them are up-sites. For each vertex in  $A_{\alpha-1,\beta+1}$  having more than two 1-neighbours and/or having a down-site outside  $A_{\alpha-1,\beta+1}$  as a 1-neighbour, we decide if that vertex is red, and whether it is an up-site or a down-site (these vertices cannot be correctly configured). We then know which of the vertices outside  $A_{\alpha-1,\beta+1}$  are correctly configured and we decide which of them are open.

This leaves a set  $W$  of vertices in  $A_{\alpha-1,\beta+1}$  with at most two neighbours which are the ones that could be correctly configured. As in (5.6), the set  $W$  has at most  $12(\beta+2)^2$  elements and for  $x$  to have a chance of being 3-pivotal there must exist a subset  $W'$  of  $W$  such that if all the vertices in  $W'$  are open and all the vertices in  $W \setminus W'$  are closed then  $x$  is 3-pivotal. So if  $Y_i < p$  for all  $x_i$  in  $W'$  and  $Y_i > p$  for all  $x_i$  in  $W \setminus W'$ , then the event  $E'_{n,3}(x)$  occurs, where  $E'_{n,3}(x)$  denotes the event that  $x$  is 3-pivotal in a modified model where the diminishments are suppressed in  $A_{\alpha-1,\beta+1}(x)$ . Hence

$$P[E'_{n,3}(x)] \geq [p(1-p)]^{12(\beta+2)^2} P_{n,3}(x).$$

Adding or removing extra non-red vertices in  $A_{\alpha,\beta}(x)$  does not affect event  $E'_{n,3}(x)$  and therefore  $E'_{n,3}(x)$  is independent of the event  $R_n(x, \alpha, \beta)$ . Also, if  $E'_{n,3}(x) \cap R_n(x, \alpha, \beta)$  occurs, then there are at most  $12(\beta+2)^2$  correctly configured red vertices in  $A_{\alpha-1,\beta+1}$ , and the probability that none of these is diminished is at least  $(1-q)^{12(\beta+2)^2}$ . In this case event  $E_{n,3} \cap R'_n(x)$  occurs, and (6.15) follows with

$$\delta_1 := [p(1-p)(1-q)]^{12(\beta+2)^2} \exp(-\pi(\beta^2 - \alpha^2)(1-p)).$$

□

**Proof of Lemma 6.9.** Assume for now that  $30.5 < |x| < n - 30.5$ . Create the full process of intensity  $\lambda$  outside the circle  $C_{30}$  around  $x$ , and the red process  $\mathcal{P}_{p\lambda,29,30}$  in the annulus  $A_{29,30}(x)$ . Assuming there are no other vertices in  $A_{29,30}(x)$ , determine which vertices outside  $C_{30}$  are diminished, but do not yet diminish any vertices inside  $C_{30}$ . Deem open all vertices that are red and have not been diminished at this stage. Let  $V$  be the set of vertices now connected (by an open path) to  $B_{0,2}$  and let  $T$  be those vertices path-connected to  $\partial B_n$ . Let  $S$  be the remaining open vertices, and let  $\mathcal{E}$  be the edges on  $S$  inherited from the original random connection model. Set  $\mathbf{S} = (S, \mathcal{E})$ .

Now create the red process  $\mathcal{P}_{p\lambda, 20, 29}$ . Let events  $H := H(V, T, \mathbf{S})$  and  $H' := H'(V, T, \mathbf{S})$  be as described just before Lemma 6.1 (with  $\mu = p\lambda$ ).

Event  $H$  must happen if  $E_{n,3}(x) \cap R'_n(x, 20, 29)$  is to occur. Hence by (6.15),  $P[H] \geq \delta_1 P_{n,3}(x)$ , and therefore by Lemma 6.1,  $P[H'] \geq \varepsilon(p\lambda)\delta_1 P_{n,3}(x)$ .

As in the latter part of the proof of Lemma 6.8, if  $H'$  occurs we can then form little discs  $D_1, \dots, D_{30}$  forming a path in  $C_{20}(x)$  from  $y^*$  to  $x$  and  $K_1, \dots, K_{30}$  forming a path in  $C_{20}(x)$  from  $z^*$  to  $x$ . Then  $x$  will be 4-pivotal if we have exactly one red vertex in each of these discs, no other vertices in the rest of the process  $\mathcal{P}_\lambda \cap C_{30}$ , all edges along these paths are present,  $Y_0 < p$ ,  $x$  is a down-site but its neighbours are up-sites, and no vertices in  $C_{30}$  are diminished. This all occurs with probability at least

$$\delta_2(p, q) := (0.005^2 \pi \lambda p)^{60} [f(0.9)]^{62} \exp(-900\pi\lambda)(p/8)(1-q)^{12(31^2)},$$

where the last factor is a lower bound on the probability that no diminishment occurs in  $C_{30}$ , by the same argument as in the proof of (6.15). Therefore

$$P_{n,4}(x) \geq \delta_1 \delta_2 \varepsilon(p\lambda) P_{n,3}(x),$$

for  $x$  with  $30.5 < |x| < n - 30.5$ .

Now suppose  $|x| \leq 30.5$ . Create the Poisson process in  $B_n \setminus C_{40}$ , and decide which of these vertices are red. Then create the red process in the annulus  $A_{39,40}(x)$ . Assuming there are no other vertices in  $A_{39,40}(x)$ , determine which vertices outside  $C_{40}$  are diminished, but do not yet diminish any vertices inside  $C_{40}$ . Then build up the red process in  $C_{39}$  inwards towards  $x$  until a vertex  $y$  occurs in the process which is connected to  $\partial B_n$ . Let  $H_1$  be the event that such a vertex  $y$  appears at distance  $r$  between 38 and 39 from  $x$ , so  $H_1$  must occur for  $E_{n,3}(x) \cap R'_n(x, 20, 39)$  to occur.

If  $x$  is inside  $B_{0.2}$  we can choose points  $a_0$  and  $b_0$  that they are both outside  $B_{0.3}$ , at distance between 0.7 and 0.8 from  $x$  and at distance greater than 1.1 from each other. We can then choose points  $a_1, a_2, \dots, a_{100}$  such that  $|a_i - a_{i-1}| \leq 0.8$  for  $1 \leq i \leq 100$ , and  $|a_{100} - y| \leq 0.8$ , no two  $a_i$  are within 0.1 of each other, and no  $a_i$  is within 1.1 of  $x$  or  $b_0$ , or inside  $B_{0.5}$  for  $i \geq 1$ .

Define discs  $D_i = C_{0.05}(a_i)$  and  $K_j = C_{0.05}(b_j)$ . If there is exactly one red vertex in each of these discs and no vertices anywhere else in the rest of the process  $\mathcal{P}_\lambda \cap C_{40}$ , all edges along these paths are present,  $Y_0 < p$ ,  $x$  is a down-site but its neighbours are up-sites and no vertices in  $C_{30}$  are diminished then  $x$  is 4-pivotal. If  $x$  is outside  $B_{0.2}$  we choose points in a

similar way but make sure  $b_0$  connects with a path to  $B_{0.2}$ , using little discs  $K_2, K_3, \dots, K_{50}$  which are again of radius 0.05 and are at least 1.1 from the  $a_i$ . Therefore, setting

$$\delta_3 := (p/8)(0.05^2 \pi \lambda p)^{152} \exp(-1600\pi\lambda)[f(0.9)]^{152}(1-q)^{12(41^2)}$$

and using Lemma 6.10, we have for some strictly positive continuous  $\delta_4(p, q)$  that

$$P_{n,4}(x) \geq \delta_3 P[H_1] \geq \delta_3 P[E_{n,3}(x) \cap R'_n(x, 20, 39)] \geq \delta_3 \delta_4 P_{n,3}(x).$$

Now suppose  $|x| \geq n - 30.5$ . Create the Poisson process in  $B_n \setminus C_{40}$ , and decide which of these vertices are red. Then create the red process in the annulus  $A_{39,40}(x)$ . Assuming there are no other vertices in  $A_{39,40}(x)$ , determine which vertices outside  $C_{40}$  are diminished, but do not yet diminish any vertices inside  $C_{40}$ . Then build the red process in  $C_{39} \cap B_{n-0.2}$  inwards towards  $x$  until a vertex  $y$  occurs that is connected to a path of open vertices to  $B_{0.5}$  but not to  $\partial B_n$ . Let  $H_2$  be the event that such a vertex  $y$  occurs at distance  $r$  between 38 and 39 from  $x$ , and that there is no current open path from  $B_{0.5}$  to  $\partial B_n$ , so  $H_2$  must occur for  $E_{n,3}(x) \cap R'_n(x, 20, 39)$  to occur. Given this vertex  $y$  we can find discs  $D_1, D_2, \dots, D_{100}$  and  $K_1, K_2, \dots, K_{50}$  of radius 0.05 as before such that if there is exactly one red vertex in each of these discs but no other vertices in the rest of the process on  $C_{40}$ , all edges along the paths are present,  $x$  is a down-site but its neighbours are up-sites,  $Y_0 < p$ , and no vertex in  $C_{40}$  are diminished then  $x$  is 4-pivotal. Therefore in this case

$$P_{n,4}(x) \geq \delta_3 P[H_2] \geq \delta_3 P[E_{n,3}(x) \cap R'_n(x, 20, 39)] \geq \delta_3 \delta_4 P_{n,1}(x).$$

Take  $\delta(p, q) := \delta_1 \delta_2 \delta_3 \delta_4 \varepsilon(p\lambda)$ . By its construction  $\delta$  is strictly positive and continuous in  $p$  and  $q$ , and (6.14) holds for all  $x \in B_n$ , completing the proof of the lemma.  $\square$

**Proof of Theorem 4.4.** We take  $q_0 < 1$ , and fix  $f$ . We now set  $q^* = (1 + q_0)/2$ , and choose  $\lambda > \lambda_{q_0 f}$ , and consider the graph  $RCM(\lambda, f)$ . Define  $p^* := \lambda_{q_0 f}/\lambda$ , so  $p^* \in (0, 1)$ . Now by considering a small box around  $(p^*, q^*)$ , and using Lemma 6.9, (6.12), (6.13) and the analogue of Proposition 5.1, we can find  $\varepsilon > 0$  such that  $\varepsilon < \min(p^*, 1 - p^*)$  and

$$\theta(p^* + \varepsilon, q_0) \leq \theta(p^*, q^*) \leq \theta(p^* - \varepsilon, 1).$$

Now the definition of  $p^*$  implies that  $RCM((p^* + \varepsilon)\lambda, q_0 f)$  percolates. Hence, on  $RCM(\lambda, f)$  the mixed site-bond process with parameters  $(p^* + \varepsilon, q_0)$  percolates and therefore the diminished site process with parameters  $(p^* + \varepsilon, q_0)$  percolates. Thus  $\theta(p^* + \varepsilon, q_0) > 0$  and hence  $\theta(p^* - \varepsilon, 1) > 0$  which means that  $RCM((p^* - \varepsilon)\lambda, f)$  percolates so

$$\lambda_f \leq (p^* - \varepsilon)\lambda < p^*\lambda = \lambda_{q_0 f}$$

and we are done.  $\square$

**Proof of Theorem 4.5.** Since  $S_p f \equiv S_{p/q} S_q f$ , it suffices to consider the case with  $q = 1$  so that  $S_q f \equiv f$ . Define the connection function  $g(r) = f(\sqrt{p}r)$ . Then  $S_p f \equiv pg$  so by Theorem 4.4,

$$\lambda_{S_p f} > \lambda_g$$

and hence the graph  $RCM(\lambda_{S_p f}, g)$  is the realization of a supercritical random connection model. Let  $p_c^{\text{bond}}$  and  $p_c^{\text{site}}$  denote the critical values for bond, respectively site, percolation on this graph. By Theorem 4.2, we have  $p_c^{\text{bond}} > p_c^{\text{site}}$ .

Given  $p' \in (0, p)$ , we have  $p'g \equiv (p'/p)S_p f$  so that  $\lambda_{S_p f} < \lambda_{p'g}$  by Theorem 4.4, so that the graph  $RCM(\lambda_{S_p f}, p'g)$  does not percolate, and therefore  $p_c^{\text{bond}} \geq p'$ . Hence,

$$p \leq p_c^{\text{bond}} < p_c^{\text{site}}. \quad (6.16)$$

By scaling, the graph  $RCM(\lambda_{S_p f}, g)$  is equivalent to the graph  $RCM(p^{-1}\lambda_{S_p f}, f)$  so that

$$p_c^{\text{site}} = \frac{p\lambda_f}{\lambda_{S_p f}}$$

and combining this with (6.16) yields the desired inequality  $\lambda_{S_p f} < \lambda_f$ .

For the last part, observe that  $\int_0^\infty r S_p f(r) dr = \int_0^\infty r f(r) dr$  and therefore the fact that (2.3) holds as a weak inequality for  $S_p f$  implies that it holds as a strict inequality for  $f$ .  $\square$

## 6.4 Notes

The conditions needed for the proof of Theorem 4.2 do not need to be quite as strict. In fact the proof will work as long as we have the following conditions

when we rescale  $f$  so that  $\sup\{a : f(a) > 0\} = 1$ , and remembering that  $\rho = \inf\{a > 0 : f(a) < 1\}$ .

i)  $f(a) = 1$  for all  $a < \rho$ .

ii) For any  $y > \rho$  we have  $\sup\{f(a) : a \geq y\} < 1$ .

iii) For any  $y < 1$  we have  $\inf\{f(a) : a \leq y\} > 0$ .

This allows functions that go up and down a few times but are bounded away from 0 and 1 between leaving 1 and hitting 0. It still does not include connection functions that hit 0 and then increase again, such as an annulus. It would seem that Theorem 4.2 should hold for any connection function with finite support but a different proof of Lemma 6.1 would be needed.

## 7 Infinite range Random Connection Model

We now look at connection functions which are non-increasing and have infinite support. Again we take  $d = 2$  in this section but the method can be adapted for higher dimensions. For Theorem 4.3 to apply we also need  $f$  to satisfy the following conditions.

$$\inf_{r>0} \frac{f(r+1)}{f(r)} > 0; \quad (7.1)$$

$$0 < \int_0^\infty 2\pi\lambda r f(r) dr < \infty. \quad (7.2)$$

By rescaling space we can assume without loss of generality that

$$f(0.01) < 0.1 \quad (7.3)$$

From now on we will suppose that  $f$  is non-increasing and satisfies conditions (7.1), (7.2), (7.3).

### 7.1 Proof of Theorem 4.3

Again let  $B_n$  be a circle of radius  $n$  centred on 0 and let  $\overline{B_n}$  be the complement of  $B_n$ . In the case of finite support we showed that for any configuration outside a fixed circle around a point  $x$ , we had the conditional probability of  $x$  being 2-pivotal being greater or equal to a constant multiplied by the probability of  $x$  being 1-pivotal. In the case of infinite support we cannot get such an inequality for all possible configurations. Given any fixed circle  $C_r$  around  $x$  we could have a very high number of vertices outside the circle which are connected to  $\overline{B_n}$  and a very high number of vertices that are connected to  $B_{0.5}$ , (i.e. such that the probability of a vertex in the circle being connected to both sets is almost 1), but in such a way that there is no path from  $B_{0.5}$  to  $\overline{B_n}$ . Then we can make the chance of  $x$  being 2-pivotal as small as we like as it requires the vertices nearby which cause it to be correctly configured to not be connected to both  $B_{0.5}$  and  $\overline{B_n}$ , which has arbitrarily small probability for a high enough density of vertices outside  $C_r$ . However there will be a non-vanishing probability of  $x$  being 1-pivotal as we just need no other vertices inside the fixed circle, and  $x$  to be joined to both  $B_{0.5}$  and  $\overline{B_n}$ .

Therefore we use a new strategy where we build up the process outside a fixed circle and stop if the probability of  $x$  being connected to  $B_{0.5}$  or  $\overline{B_n}$  becomes high enough.

We again use the same idea of a point being 1-pivotal and 2-pivotal but we alter the enhancement slightly to make sure that it only has a local effect. A non-red vertex  $x_1$  is correctly configured if there are exactly 4 red vertices within distance 1 of  $x_1$  whose connections form a bow-tie configuration and if there are no non-red vertices within distance 2 of  $x_1$ . With probability  $q$  we then make  $x_1$  green but close all edges from  $x_1$  apart from the edges in the bow-tie.

We also alter the definitions of being pivotal. As the proof of Proposition 5.1 does not hold in this case we consider instead the sequence of events  $A_n$  where  $A_n$  is defined to be the event that if we make all vertices outside  $B_n$  red there is a coloured path from  $B_{0.5}$  to  $\overline{B_n}$ . Then  $A_n \supset A_{n+1}$  for all  $n$ . We let  $\psi_n(p, q) = P[A_n]$  and we let  $\psi(p, q) = \lim_{n \rightarrow \infty} (\psi_n(p, q))$ . We then have the following result.

**Proposition 7.1** *There is a.s. an infinite connected component using only coloured vertices in the original process if and only if  $\psi(p, q) > 0$*

**Proof.** Let  $A_\infty$  be the event that there is an infinite coloured component intersecting  $B_{0.5}$ . It is easy to see that  $A_\infty \subset \cap_{n=1}^\infty A_n$ . But also, if  $A_\infty$  does not occur and all vertices have finite degree in the RCM, then  $A_n$  does not occur for large enough  $n$  (namely, for  $n$  such that there is no link in the RCM from the union of coloured components intersecting  $B_{0.5}$  to the complement of  $B_{n-1}$ ). Therefore, since all vertices have finite degree a.s.,  $P[A_\infty] = \lim_{n \rightarrow \infty} (P[A_n]) = \psi(p, q)$ .  $\square$

The following lemma is the equivalent of Lemma 5.1.

**Proposition 7.2** *For all  $n > 0.5$  and  $p \in (0, 1)$  and  $q \in (0, 1)$  it is the case that*

$$\frac{\partial \psi_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,1}(x, p, q) \, dx \quad (7.4)$$

and

$$\frac{\partial \psi_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,2}(x, p, q) \, dx. \quad (7.5)$$

It is easy to see that the proof of Lemma 5.1 holds for this case as well, just by replacing  $\theta_n$  by  $\psi_n$  in the proof. We now just need the equivalent of Lemma 5.2 for this case, which is given below.

**Lemma 7.1** *Suppose the random connection function  $f$  is non-increasing and satisfies conditions (7.1), (7.2), (7.3), then there exists a function  $\delta(p, q)$  such that for all  $n > 100$  and for all  $x \in B_n$ ,*

$$P_{n,2}(x) > \delta(p, q)P_{n,1}(x)$$

where  $\delta$  is independent of  $n$  and  $x$ , and is strictly positive and continuous on  $(0, 1)^2$ .

Using Proposition 7.1, Lemma 5.1 and Lemma 7.1 we can argue as before to prove Theorem 4.3.

**Start of proof of Lemma 7.1.** Choose  $c \in (0, f(25))$  such that  $f(r+1) \geq cf(r)$  for all  $r > 0$ . Let  $k$  be the expected degree of a vertex in  $C$ , so

$$k = \int_0^\infty 2\pi\lambda r f(r) dr. \quad (7.6)$$

For any point  $y$  let  $C_r(y)$  be a circle of radius  $r$  centred on  $y$ . Fix  $x$  and write  $C_r$  for  $C_r(x)$ . Again for any set of vertices  $S$  and any point  $z$  let  $p(z, S)$  be the probability that a vertex at  $z$  is joined to at least one of the vertices in  $S$ . So  $1 - p(z, S) = \prod_{s \in S} (1 - f(|z - s|))$ . The following propositions are used in the proof of Lemma 7.1.

**Proposition 7.3** *Suppose  $c$  is as above. Suppose  $d \in \mathbb{Z}$ . Then for any points  $x$  and  $y$  in  $\mathbb{R}^2$  such that  $|x - y| \leq d$  and any set of vertices  $M$ ;*

$$p(y, M) \geq c^d p(x, M).$$

**Proof.** We prove it by induction on the size of  $M$ . For any point  $y$  within  $d$  of  $x$  and any set  $M$  we have that  $f(|y - m|) \geq c^d f(|x - m|)$  for all  $m \in M$  so it is true for  $|M| = 1$ .

Let  $M = M' \cup m$  so we assume

$$1 - p(y, M') \leq 1 - c^d p(x, M').$$



Then,

$$\begin{aligned}
1 - p(y, M) &= (1 - p(y, M'))(1 - p(y, m)) \\
&\leq (1 - c^d p(x, M'))(1 - c^d p(x, m)) \\
&= 1 - c^d p(x, M') - c^d p(x, m) + c^{2d} p(x, M') p(x, m) \\
&\leq 1 - c^d [p(x, M') + p(x, m) - p(x, m) p(x, M')] \\
&= 1 - c^d p(x, M)
\end{aligned}$$

so  $p(y, M) \geq c^d p(x, M)$ . □

**Proposition 7.4** *Suppose the random connection function  $f$  is non-increasing and satisfies conditions (7.1), (7.2), (7.3). Let  $d \in \mathbb{Z}$  and let  $z$  be a point in  $\mathbb{R}^2$ . Let  $\lambda$  be fixed and let  $k$  be fixed as in (7.6). Suppose a set of vertices  $V$  satisfies  $|V \cap D| \leq d$  for all regions  $D \in \mathbb{R}^2$  of diameter 2 and  $|V \cap C_6(z)| = 0$ . Then*

$$p(z, V) < 1 - \kappa(d)$$

where we set  $\kappa(d) := e^{-6d \frac{k}{\lambda}}$ .

**Proof.** In any subset of the plane of diameter less than 2 there are at most  $d$  vertices in  $V$ . Therefore in any annulus  $C_{r+1}(z) \setminus C_r(z)$ , with  $r \geq 6$  we can split the annulus up into sectors and end up with no more than  $3\pi r$  regions with diameter less than 2. Therefore in the region  $C_{r+1}(z) \setminus C_r(z)$  there are no more than  $(3\pi r)d$  vertices in  $V$ . There are no elements of  $V$  inside  $C_6(z)$ . We also have that  $1 - x \geq e^{-2x}$  for  $x < 0.1$  and note that  $f(r) < 0.1$  for  $r > 6$  (as  $f(0.01) < 0.1$ ). Therefore,

$$\begin{aligned}
1 - p(y, V) &\geq \prod_{i \in V} (1 - f(|y - i|)) \\
&\geq \prod_{r \geq 6} [1 - f(r)]^{3\pi r d} \\
&\geq \prod_{r \geq 6} \exp(-6\pi r f(r) d) \\
&= \exp\left(-\sum_{r \geq 6} 6\pi r f(r) d\right) \\
&\geq \exp\left(-\int_0^\infty 12\pi r f(r) d dr\right) = e^{-6d \frac{k}{\lambda}}.
\end{aligned}$$

□

**Proof of Lemma 7.1 continued.**

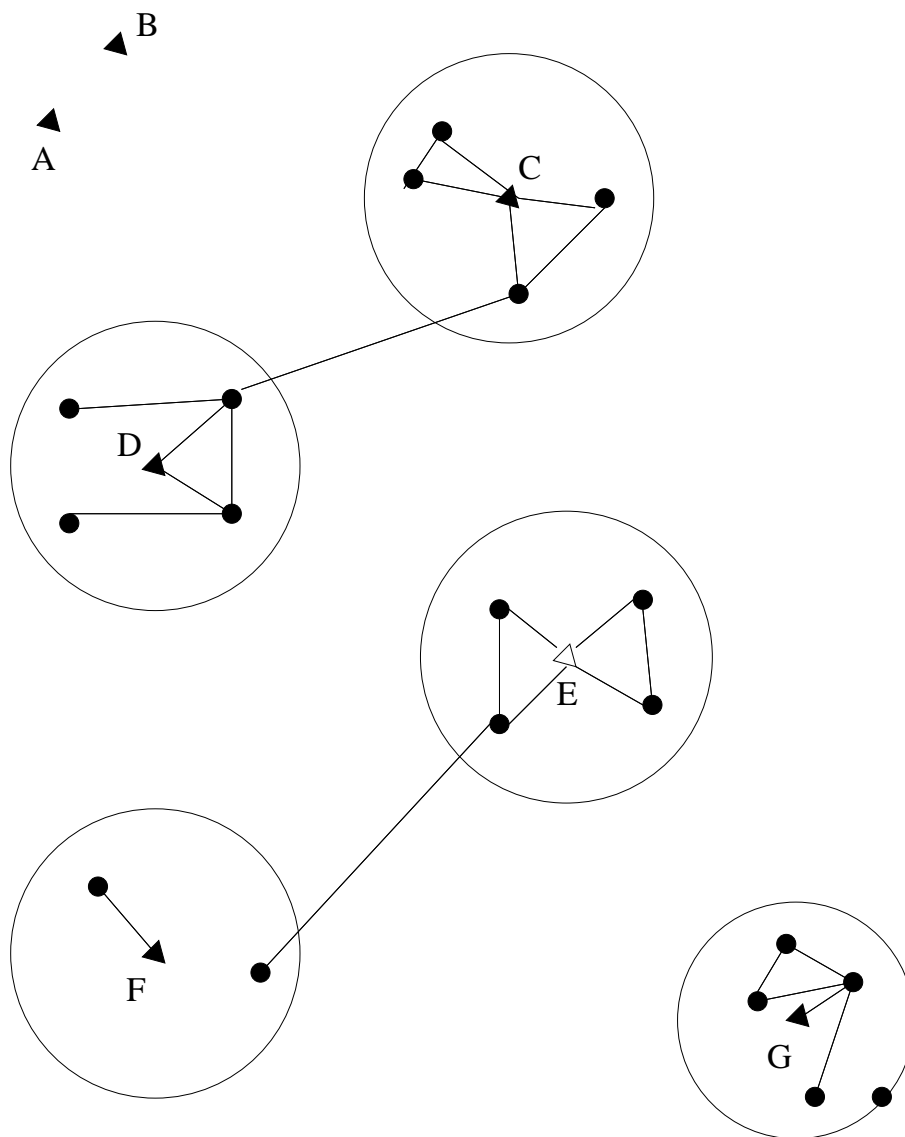


Figure 7.1: This shows some of the process after stage 1. The solid triangles are closed vertices. The unsolid triangle is a green vertex and the solid circles are red vertices. Bonds between vertices are shown by lines. Out of the non-red vertices only C and E are correctly configured, and only E has been enhanced. The circles around the non-red vertices are the only areas where the red process has been built up.

**If  $x$  is not near the origin or the edge of  $B_n$ .** If  $15 < |x| < n - 15$  then we start by building up the process outside  $C_{12}$  in stages as follows.

**Stage 1.** We start off by creating the process of non-red vertices inside  $B_n \setminus C_{12}$  (see Figure 7.1), i.e. a process with intensity  $\lambda(1 - p)$  on the whole of  $B_n \setminus C_{12}$ . We then let  $U$  be the set of vertices that have no other vertices in their 2-neighbourhood. For every vertex  $u \in U \cap C_{14}$  we create the set of non-red vertices inside  $C_2(u) \cap C_{12}$ . Any vertices that now do have a vertex in their 2-neighbourhood are removed from  $U$ . For every vertex  $y$  in  $U$  (starting from the origin and moving outwards) we then build up the process of red vertices outwards from  $y$  in the 1-neighbourhood of  $y$  (with intensity  $p\lambda$  inside  $B_n$  and  $\lambda$  outside  $B_n$ ), but stop if we get 5 vertices. We let  $H_y$  be the region in which we have built up the process. We put in the edges involving the vertices in  $H_y$  and the edges to  $y$ . If  $H_y$  is the 1-neighbourhood of  $y$  and we have 4 red vertices in  $H_y$  and  $y$  is correctly configured then we put  $y$  in a set  $U'$  with probability  $q$ . If not then we do not. We then move on to the next vertex in  $U$  until we have done them all, and we add edges amongst all the vertices that have occurred so far.  $U'$  is the complete set of green vertices in  $B_n \setminus C_{12}$ . Note that all the  $H_y$ 's are disjoint. We let  $H = \cup_{y \in U} H_y$ , so we have built up the process of red vertices in  $H$  so far but nowhere else. We let  $T$  be the possibly empty set of red vertices connected via a coloured path to  $B_{0.5}$ .

**Stage 2.** In this stage we build up our starting set of red vertices that are connected to  $B_{0.5}$ . We build up the rest of the process of red vertices outwards from the origin in  $B_{0.5}$ , adding vertices to  $T$  (with intensity  $p\lambda$  on  $B_{0.5}$  but ignoring where we have looked already, i.e. the region  $H$ ). Again we add in edges as we go, and we stop if  $p(x, T) > c^2$  (in which case we go to Step 6 of Stage 3) or if we have built up the whole process in  $B_{0.5}$ . We then let  $T_0$  be the set of red vertices connected via a coloured path to  $B_{0.5}$  (this is the same as the set  $T$  for now but  $T_0$  stays fixed, whilst we add vertices to  $T$  at the next stage). We let  $W$  be the possibly empty set of red vertices currently connected to  $\overline{B_n}$ . We let  $V_0$  be the set of red vertices not in either  $T_0$  or  $W$ . In any square of side 1 there is at most 1 vertex in  $U$ . So in any square of side 6 there are at most 36 vertices in  $U$ . Each of these corresponds to at most 5 vertices in  $V_0$ . Therefore in any region  $D$  of diameter 2 we have the following bound;

$$|V_0 \cap D| \leq 180. \quad (7.7)$$

**Stage 3.** The next stage is to build up the set  $T$  generation by generation from our starting point of  $T_0$ , stopping if  $p(x, T)$  becomes high enough. We add in vertices one by one that are connected to the latest generation of  $T$ , stopping if the probability of  $x$  being joined to  $T$  becomes high enough. As we add in vertices one by one, this probability can never jump by too much so we also have an upper bound on the probability of  $x$  being joined to  $T$ . The algorithm goes as follows:

Step 1) Let  $j = 0$

Step 2) Let  $k = 0$  and  $|y_0| = 0.5$

Step 3) We build radially symmetrically outwards from  $B_{|y_k|}$  with intensity  $p\lambda p(z, T_j)[1 - p(z, T \setminus T_j)]\mathbf{1}_{z \in B_n \setminus (B_{0.5} \cup C_{12} \cup H)} dz$  either until we get a vertex  $y_{k+1}$  or until we exhaust  $B_n$ . If we do not get a vertex then we go to Step 5.

Step 4) We add  $y_{k+1}$  to  $T_{j+1}$  and add in all edges involving  $y_{k+1}$  conditional on there being at least one edge to the set  $T_j$  and no edges to the set  $T \setminus T_j$ . Any vertices in  $V_j$  that are now connected via a coloured path to  $B_{0.5}$  we add to  $T_{j+1}$ . If we now have  $p(x, T \cup T_{j+1}) > c^2$  then we go to Step 6; if not we increase  $k$  by one and repeat Step 3.

Step 5) If  $T_{j+1} = \emptyset$  we go to Step 7. If not then we add  $T_{j+1}$  to  $T$  and we let  $V_{j+1} = V_j \setminus T_{j+1}$ . We increase  $j$  by 1 and go back to Step 2.

Step 6) We say the event  $F_1$  has occurred. We add  $T_{j+1}$  to  $T$ , we let  $t'$  be  $y_{k+1}$  and we stop.

Step 7) The event  $F_1$  has not occurred.

**Stage 4.** We then build up the set  $W$  of vertices that are connected to  $\overline{B_n}$  in similar fashion. This stage is the analogue of Stage 2 as we build up the starting point for  $W$ . We build outwards from  $B_n$  with intensity  $\lambda$ , stopping if we get  $p(x, W) > c^2$  (in which case we then go to Step 6 of Stage 5). We then let  $W_0$  be the set of vertices connected by a coloured path to  $\overline{B_n}$ . We let  $Y_0 = V_0 \setminus (W_0 \cup T)$ .

**Stage 5.** This is the analogue of Stage 3 as we build up  $W$  generation by generation. The algorithm is as follows.

Step 1) Let  $j = 0$ ,

Step 2) Let  $k = 0$  and  $|x_0| = 0.5$

Step 3) We build radially symmetrically outwards from  $B_{|x_k|}$  with intensity  $p\lambda p(z, W_j)[1 - p(z, W \setminus W_j)][1 - p(z, T)]I_{z \in B_n \setminus (B_{0.5} \cup C_{12} \cup H)} dz$  either until

we get a vertex  $x_{k+1}$  or until we exhaust  $B_n$ . If we do not get a vertex then we go to Step 5.

Step 4) We add  $x_{k+1}$  to  $W_{j+1}$  and add in edges to  $x_{k+1}$  appropriately. Any vertices in  $Y_j$  that are now connected via a coloured path to  $\overline{B_n}$  we add to  $W_{j+1}$ . If we now have  $p(x, (W \cup W_{j+1})) > c^2$  then we go to Step 6; if not we increase  $k$  by one and repeat Step 3.

Step 5) If  $W_{j+1} = \emptyset$  we go to Step 7. If not then we add  $W_{j+1}$  to  $W$  and we let  $Y_{j+1} = Y_j \setminus W_{j+1}$ . We increase  $j$  by 1 and go back to Step 2.

Step 6) We say the event  $F_2$  has occurred. We add  $W_{j+1}$  to  $W$ , we let  $w'$  be  $x_{k+1}$  and we stop.

Step 7) The event  $F_2$  has not occurred.

**Stage 6.** We now build up the rest of the process outside  $C_{12}$ . We let  $T'$  be the latest generation of  $T$  (which may be empty) and let  $T''$  be the generation before that. We let  $y'$  be the outermost vertex in  $T'$  if there was one, or else some point on the edge of  $B_n$ . Similarly we define  $W'$  and  $W''$  and  $x'$ . So any vertices that appear in the rest of the process cannot be joined to  $T \setminus (T' \cup T'')$  but can be joined to  $T'$  and may be able to be joined to  $T''$  if they are further out than  $y'$ . Therefore we create the rest of the red vertices outside  $C_{12}$  as follows. It is an inhomogenous Poisson Process with intensity  $p\lambda I_{z \in B_n \setminus (B_{0.5} \cup C_{12} \cup H)} g(|z|) h(|z|)$  where

$$g(r) = [1 - p(z, T \setminus (T' \cup T''))], \quad \text{for } r > |y'|$$

and

$$g(r) = [1 - p(z, T \setminus T'')] \quad \text{for } r < |y'|.$$

Similarly,

$$h(r) = [1 - p(z, W \setminus (W' \cup W''))] \quad \text{for } r > |x'|$$

and

$$h(r) = [1 - p(z, W \setminus W'')] \quad \text{for } r < |x'|.$$

We then add in all edges involving these new vertices, conditional on having no edges to  $T$  and  $W$  that should not be there. We may also have to finish building the process of red vertices inside  $B_{0.5}$  or outside  $B_n$ . We let  $E_1$  be the event that there is no current path from  $B_{0.5}$  to  $\overline{B_n}$ , so this must occur for  $x$  to be pivotal. We let  $Q$  be the region we have not built up the process in yet, i.e.  $C_{12} \setminus H$  (where  $H$  is as defined in Stage 1). We let  $V'$  be

the set of red vertices that were in  $V_0$  (as defined in Stage 2) and are not in  $T$  or  $W$ . We let  $S$  be the set of red vertices not in  $T$ ,  $V'$  or  $W$ , which are the vertices that occurred in Stage 6.

Let  $\chi$  be the configuration of everything that has occurred by the end of Stage 5, i.e. all the vertices except those in  $S$  and whether they are green, red or closed and the edges between them. The following Proposition will complete the proof of Lemma 7.1 for  $x$  not near  $B_{0.5}$  or the edge of  $B_n$ .

**Proposition 7.5** *There exists a function  $\gamma_1(p, q)$  which is strictly positive and continuous on  $(0, 1)^2$  such that for all  $x : 15 < |x| < n - 15$  and for any possible configuration  $\chi$  the conditional probabilities of  $x$  being 1-pivotal and 2-pivotal satisfy*

$$P[E_{n,2}(x)|\chi] \geq \gamma_1 P[E_{n,1}(x)|\chi].$$

We have 4 cases:

Case 1; Both  $F_1$  and  $F_2$  occur. We have that  $p(x, T) > c^2$  and  $p(x, W) > c^2$ . We also have  $p(x, T \setminus (V_0 \cup t')) < c^2$  (where  $t'$  is as defined in Stage 3 Step 6) as otherwise the algorithm would have stopped earlier.

We let  $A_1, A_2, A_3, A_4$  be circles of radius 0.1 centred on the points with coordinates  $(\pm 0.5, \pm 0.5)$  in relation to  $x$ . Now for any point  $y$  within 1 of  $x$  and taking  $M$  to be the set  $T \setminus (V_0 \cup t')$  we can apply Proposition 7.3 to get  $p(y, M) \leq c^{-1}p(x, M) \leq c^{-1}c^2 < 0.1$ . Also  $p(y, t') < 0.1$  and by applying Proposition 7.4 to  $V_0$  and using (7.7) we get  $p(y, V_0) < 1 - \kappa(180)$ . Therefore for any  $y$  in  $C_1$ ,

$$\begin{aligned} 1 - p(y, T) &= [1 - p(y, t')][1 - p(y, V_0)][1 - p(y, T \setminus (V_0 \cup t'))] \\ &> 0.9^2 \kappa(180) \\ &=: \delta_1. \end{aligned}$$

Similarly  $1 - p(y, W) > \delta_1$  for any  $y$  in  $C_1$ .

Taking  $M$  to be the set  $T$  we have  $p(y, T) \geq cp(x, T) > c^3 := \delta_2$ . Similarly  $p(y, W) > \delta_2$ .

Therefore if we have exactly one red vertex in  $A_1$  it will be connected to  $T$  but not  $W$  or  $V'$  with probability at least  $\delta_1[\kappa(180)]\delta_2$ . Let  $G$  be the following event:

- 1) We get exactly one red vertex in each of  $A_1, \dots, A_4$ , which we call  $b_1, \dots, b_4$ .
- 2) The vertex  $b_1$  connects to at least one vertex in  $T$  but no vertices outside  $Q$  that are not in  $T$ ,  $b_3$  connects to at least one vertex in  $W$  but no

vertices outside  $Q$  that are not in  $W$  and  $b_2$  and  $b_4$  connect to no vertices outside  $Q$ .

3) There are edges  $b_1 \sim b_2$  and  $b_3 \sim b_4$  and all the  $b_i$ 's connect with  $x$  but there are no other edges amongst them.

4) There are no other vertices inside  $Q$ .

5) The uniform random variable associated with  $x$  satisfies  $Y_0 > p$ .

The conditional probability that  $G$  occurs is at least

$$\begin{aligned} & [(0.1^2 \pi \lambda p \exp(-0.1^2 \pi \lambda p))]^4 \delta_1^6 \delta_2^2 [\kappa(180)]^4 \\ & \times f(1)^6 (1 - f(0.5))^4 \exp(-(12)^2 \pi \lambda) (1 - p) =: \delta_3. \end{aligned}$$

We now split up stage 6 of the algorithm into two processes, namely the process  $S_2$  of vertices in  $S$  that are joined to one of  $b_1, \dots, b_4$  and the process  $S_1$  of those that are not. The number of vertices in  $S_2$  has a Poisson distribution with mean less than  $4k$  as  $k$  is the mean number of neighbours of a vertex in an infinite process. Let  $E'_1$  be the event that the vertices in  $S_1$  do not complete a path from  $B_{0.5}$  to  $\overline{B}_n$ . Let  $E_2$  be the event that no vertices occur in the  $S_2$  process. Therefore the events  $E'_1$  and  $E_2$  are independent and  $P[E_2] > \exp(-4k)$ . The intensity of the  $S_1$  process is everywhere less than the intensity of the whole  $S$  process so the probability of  $E'_1$  occurring is greater than the probability of  $E_1$  occurring as they are decreasing events (where  $E_1$  is as described in Stage 6).

If the events  $F_1 \cap F_2 \cap G \cap E'_1 \cap E_2$  occur then  $x$  is 2-pivotal. Therefore for any configuration  $\chi$  where  $F_1 \cap F_2$  occurs

$$\begin{aligned} P[E_{n,2}(x)|\chi] & \geq \delta_3 \exp(-4k) P[E_1|\chi] \\ & \geq \delta_3 \exp(-4k) P[E_{n,1}|\chi] \\ & =: \varepsilon_1 P[E_{n,1}|\chi] \end{aligned}$$

Case 2; Both  $F_1$  and  $F_2$  do not occur. This means that  $p(x, T) < c^2$  and  $p(x, W) < c^2$  and the process of red vertices connected to  $T \cup W$  is complete outside  $Q = C_{12} \setminus H$ . Therefore for  $x$  to be pivotal one of the following events must occur:

1) The vertex  $x$  is joined to  $T$  and  $W$ .

2) The vertex  $x$  is joined to  $T$  and some vertex occurs in the red process inside  $Q$  that connects to  $W$  but not  $T$ . (Note that the vertex has to be in

the red process as a green vertex can only connect to red vertices within unit distance but there are no red vertices in  $W$  within 13 of  $x$  as  $p(x, W) < c^2 < f(13)$ .)

3) The vertex  $x$  is joined to  $W$  and some vertex occurs in the red process inside  $Q$  that connects to  $T$  but not  $W$ .

4) Some vertex occurs in the red process inside  $Q$  that connects to  $W$  but not  $T$  and another vertex occurs that connects to  $T$  but not  $W$ .

The process of red vertices inside  $Q$  that connect to  $T$  but not  $W$  is an inhomogeneous Poisson process with intensity  $p\lambda p(z, T)(1 - p(z, W))$ . But for any  $z$  inside  $Q$  we have  $|z - x| \leq 12$  so by Proposition 7.3 this means that  $p(x, T) \geq c^{12}p(z, T)$ . Therefore the probability of a vertex occurring in this process is less than the probability of a vertex occurring in a homogeneous process on  $Q$  with intensity  $p\lambda p(x, T)c^{-12}$ . The number of vertices in this process has a Poisson distribution with mean no more than  $Kp(x, T)$  where  $K := 144p\lambda c^{-12}\pi$ . So the probability that a vertex occurs is no more than  $Kp(x, T)$  by Markov's inequality. The process of red vertices inside  $Q$  that connect to  $W$  but not  $T$  is an independent inhomogeneous Poisson process with intensity  $p\lambda p(z, W)(1 - p(z, T))$  and by the same argument the probability that a vertex occurs is no more than  $Kp(x, T)$ . Therefore the probability of one of the four events above happening is no more than  $(1 + K)^2p(x, T)p(x, W)$  so given a configuration  $\chi$  where  $F_1^c \cap F_2^c$  occurs the conditional probability of  $x$  being 1-pivotal satisfies;

$$P[E_{n,1}(x)|\chi] \leq (1 + K)^2p(x, T)p(x, W)P[E_1|\chi].$$

Again we let  $A_1, A_2, A_3, A_4$  be circles of radius 0.1 centred on the points with coordinates  $(\pm 0.5, \pm 0.5)$  in relation to  $x$ .

Now for any point  $y$  within 1 of  $x$  again we have  $p(y, T) \geq cp(x, T)$  and we have  $1 - p(y, W) \geq \delta_1$ .

Therefore if we have exactly one red vertex in  $A_1$  it will be connected to  $T$  but not  $W$  or  $V'$  with probability at least  $\delta_1[\kappa(180)]cp(x, T)$ .

Similarly the probability that a vertex in  $A_3$  is connected to  $W$  but not  $T$  or  $V'$  is at least  $\delta_1[\kappa(180)]cp(x, W)$ .

Let  $G$  be the same event as before.

The conditional probability that  $G$  occurs is at least

$$[(0.1^2\pi\lambda p \exp(-0.1^2\pi\lambda p))^4\delta_1^6[\kappa(180)]^4c^2p(x, T)p(x, W)]$$



$$\times f(1)^6(1 - f(0.5))^4 \exp(-(12)^2\pi\lambda)(1 - p) =: \delta_4 p(x, T)p(x, W).$$

Now we carry out Stage 6 of the algorithm and let  $E'_1$  and  $E_2$  be as before. Again  $P[E'_1|\chi] \geq P[E_1|\chi]$  and  $P[E_2] > \exp(-4k)$ . If the events  $F_1^c \cap F_2^c \cap G \cap E'_1 \cap E_2$  occur then  $x$  is 2-pivotal. Therefore for any configuration  $\chi$  where  $F_1^c \cap F_2^c$  occurs

$$\begin{aligned} P[E_{n,2}(x)|\chi] &\geq \delta_4 p(x, T)p(x, W) \exp(-4k) P[E_1|\chi] \\ &\geq \frac{\delta_4 \exp(-4k)}{(1 + K)^2} P[E_{n,1}|\chi] \\ &=: \varepsilon_2 P[E_{n,1}|\chi]. \end{aligned}$$

Case 3; The event  $F_1$  occurs but  $F_2$  does not. This means that for  $x$  to be pivotal one of the following events must occur.

- 1) The vertex  $x$  is joined to  $W$ .
- 2) Some vertex occurs in the red process inside  $Q$  that connects to  $W$  but not  $T$ .

By the same argument as for Case 2 the probability of one of the two events above happening is no more than  $(1 + K)p(x, W)$  so given a configuration  $\chi$  where  $F_1 \cap F_2^c$  occurs the conditional probability of  $x$  being 1-pivotal satisfies;

$$P[E_{n,1}(x)|\chi] \leq (1 + K)p(x, W)P[E_1|\chi].$$

Let  $G$  be the same event as before. By similar arguments to the previous 2 cases the conditional probability that  $G$  occurs is at least

$$\begin{aligned} &[(0.1^2\pi\lambda p \exp(-0.1^2\pi\lambda p))^4 \delta_1^6[\kappa(180)]^4 cp(x, W)\delta_2 \\ &\times f(1)^6(1 - f(0.5))^4 \exp(-(12)^2\pi\lambda)(1 - p) =: \delta_5 p(x, W). \end{aligned}$$

Now we carry out Stage 6 of the algorithm and let  $E'_1$  and  $E_2$  be as before. Again  $P[E'_1|\chi] \geq P[E_1|\chi]$  and  $P[E_2] > \exp(-4k)$ . If the events  $F_1 \cap F_2^c \cap G \cap E'_1 \cap E_2$  occur then  $x$  is 2-pivotal. Therefore for any configuration  $\chi$  where  $F_1 \cap F_2^c$  occurs

$$\begin{aligned}
P[E_{n,2}(x)|\chi] &\geq \delta_5 p(x, W) \exp(-4k) P[E_1|\chi] \\
&\geq \frac{\delta_5 \exp(-4k)}{1+K} P[E_{n,1}|\chi] \\
&=: \varepsilon_3 P[E_{n,1}|\chi].
\end{aligned}$$

Case 4; The event  $F_2$  occurs but  $F_1$  does not. The same arguments as Case 3 but with the roles of  $T$  and  $W$  reversed leads to the following equation for these configurations,

$$P[E_{n,2}(x)|\chi] \geq \varepsilon_3 P[E_{n,1}|\chi].$$

Therefore if we take  $\gamma_1 := \varepsilon_1 \varepsilon_2 \varepsilon_3$  the proof of Proposition 7.5 is complete.

□

This shows that if  $x$  is not near  $B_{0,5}$  or the edge of  $B_n$  then

$$P_{n,2}(x) > \gamma_1(p, q) P_{n,1}(x).$$

**If  $x$  is near the origin.** If  $|x| \leq 15$  then we start off by building up the process outside  $C_{23}$  in stages. Firstly we repeat Stage 1 of the last section except  $C_{12}$  is replaced by  $C_{23}$ . This creates all of the green vertices inside  $B_n \setminus C_{23}$  and builds up the red process on a set  $H$ . Let  $Q = C_{23} \setminus H$ . We let  $\mu_1 := c^{17}$ . We let  $W$  be the possibly empty set of red vertices currently connected to  $\overline{B_n}$ . We let  $V_0$  be the set of red vertices not in  $W$ . There is no  $T$  this time as  $B_{0,5}$  is contained in  $C_{16}$  so there is no equivalent of Stages 2 and 3. Next we build up the set  $W_0$  of vertices that are initially connected to  $\overline{B_n}$  as in Stage 4 and then the set  $W$  of vertices connected to  $\overline{B_n}$  as in Stage 5 but this time we stop if we get  $p(x, W) > \mu_1$ . Define  $F'_2$  to be the event that the algorithm stops with  $p(x, W) > \mu_1$ , so this is the equivalent of  $F_2$ . Then build up the rest of the process outside  $C_{23}$  as in Stage 6. Let  $S$  be the set of vertices that occur in Stage 6. Let  $\chi$  be the configuration of everything that has occurred by the end of Stage 5, i.e. all the vertices except those in  $S$  and whether they are green, red or closed and the edges between them. The following Proposition will complete the proof of Lemma 7.1 for  $x$  near  $B_{0,5}$  and is the equivalent of Proposition 7.5.

**Proposition 7.6** *There exists a function  $\gamma_2(p, q)$  which is strictly positive and continuous on  $(0, 1)^2$  such that for all  $x$  with  $|x| < 15$  and for any possible configuration  $\chi$  the conditional probabilities of  $x$  being 1-pivotal and 2-pivotal satisfy*

$$P[E_{n,2}(x)|\chi] \geq \gamma_2 P[E_{n,1}(x)|\chi].$$

We have 2 cases:

Case 1;  $F'_2$  occurs. Then  $p(x, W) > \mu_1$ . Let  $M$  be the set  $W \setminus (V_0 \cup w')$ . Then  $p(x, M) < \mu_1$ . Therefore for any point  $z$  within 16 of  $x$ , by Proposition 7.3 we have  $p(z, M) < c^{-16}\mu_1 < 0.1$ . Also  $p(z, w') < 0.1$  and  $p(z, V_0) < 1 - \kappa(180)$  so  $1 - p(z, W) > 0.9^2\kappa(180) = \delta_1$ . As before for any point  $y$  within 1 of  $x$  we have  $p(y, W) \geq cp(x, W) > c\mu_1 =: \delta_6$ .

If  $0.7 < |x| < 15$  then we let  $A_1, A_2, A_3, A_4$  be circles of radius 0.01 centred on the points with coordinates  $(\pm 0.3, \pm 0.2)$  in relation to  $x$  with the line from the origin to  $x$  as the  $y$ -axis (see Figure 7.2). Let  $A_5$  be a circle of radius 0.01 centred on the point at distance 0.4 from the origin directly opposite  $x$ .

Let  $H_1$  be the following event:

- 1) We get exactly one red vertex in each of  $A_1, \dots, A_5$ , which we call  $b_1, \dots, b_5$  respectively.
- 2) The vertex  $b_1$  connects to at least one vertex in  $W$  but no vertices outside  $Q$  that are not in  $W$  and  $b_2, b_3, b_4, b_5$  connect to no vertices outside  $Q$ .
- 3) There are edges  $b_1 \sim b_2$ ,  $b_3 \sim b_5$  and  $b_3 \sim b_4$  and all of  $b_1, b_2, b_3, b_4$  connect with  $x$  but there are no other edges amongst  $b_1, \dots, b_5, x$ .
- 4) There are no other vertices inside  $Q$ .
- 5) The uniform random variable associated with  $x$  satisfies  $Y_0 > p$ .

The conditional probability that  $H_1$  occurs is at least

$$\begin{aligned} & [(0.01^2\pi\lambda p \exp(-0.01^2\pi\lambda p))^5 \delta_1^4 \delta_6 [\kappa(180)]^5 \\ & \times f(1)^6 f(16)(1 - f(0.1))^8 \exp(-(23)^2\pi\lambda)(1 - p) =: \delta_7. \end{aligned}$$

If  $|x| < 0.7$  then we let  $A_1$  and  $A_2$  be circles of radius 0.01 centred  $\pm(0.01, 0.01)$  so these are contained in  $B_{0.5}$ . Let  $A_3$  and  $A_4$  be disjoint circles of radius 0.01 centred on points at distance 0.8 from  $x$  in such a way that these do not intersect  $B_{0.5}$  (see Figure 7.3).

Let  $H_1$  be the following event:

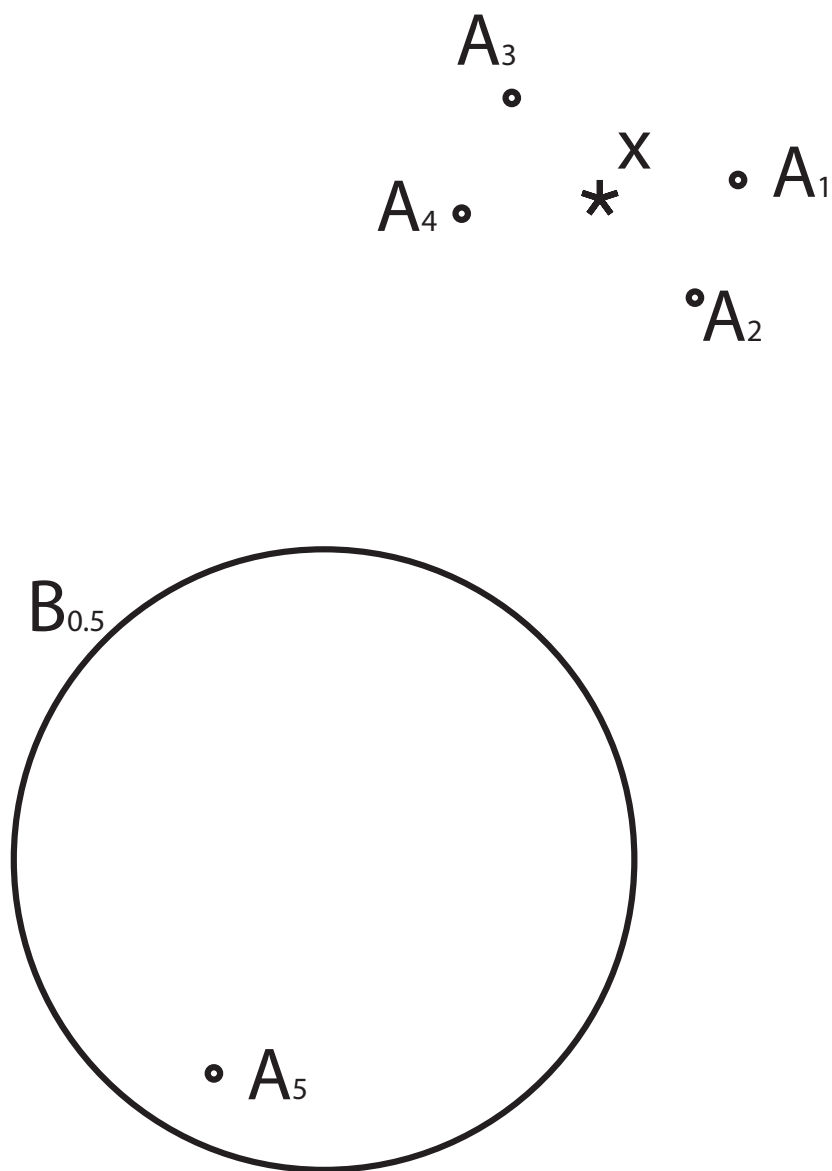


Figure 7.2: For the case  $0.7 < |x| < 15$ .

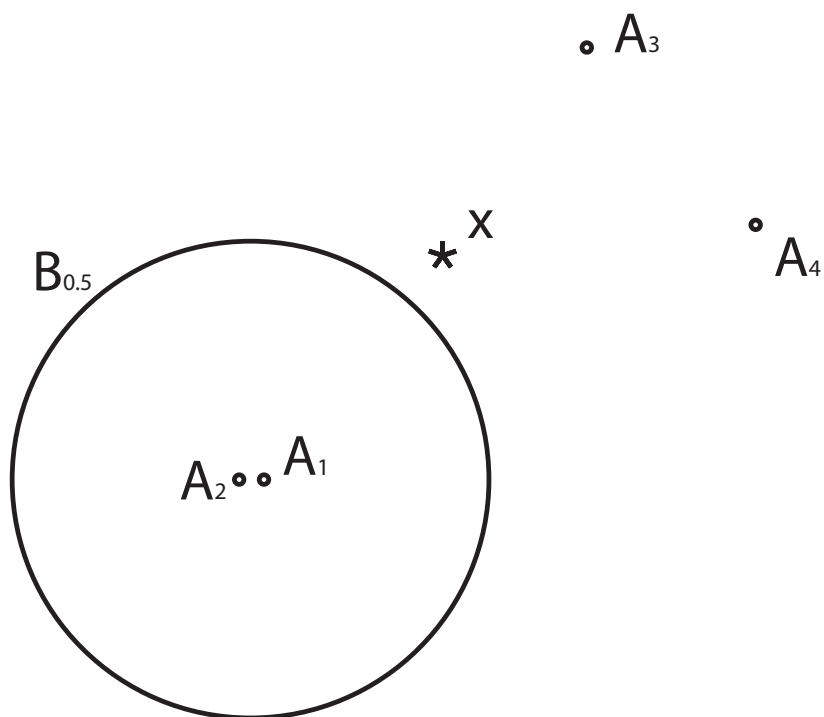


Figure 7.3: For the case  $|x| < 0.7$ .

- 1) We get exactly one red vertex in each of  $A_1, \dots, A_4$ , which we call  $b_1, \dots, b_4$  respectively.
  - 2) The vertex  $b_3$  connects to at least one vertex in  $W$  but no vertices outside  $Q$  that are not in  $W$  and  $b_1, b_2, b_4$  connect to no vertices outside  $Q$ .
  - 3) There are edges  $b_1 \sim b_2$  and  $b_3 \sim b_4$  and all of  $b_1, b_2, b_3, b_4$  connect with  $x$  but there are no other edges amongst  $b_1, \dots, b_4, x$ .
  - 4) There are no other vertices inside  $Q$ .
  - 5) The uniform random variable associated with  $x$  satisfies  $Y_0 > p$ .
- The conditional probability that  $H_1$  occurs is at least

$$\begin{aligned} & [(0.01^2 \pi \lambda p \exp(-0.01^2 \pi \lambda p))^4 \delta_1^3 \delta_6 [\kappa(180)]^4 \\ & \times f(2)^6 (1 - f(0.1))^4 \exp(-(23)^2 \pi \lambda) (1 - p) =: \delta_8. \end{aligned}$$

Now split up Stage 6 of the algorithm into two processes as before, namely the process  $S_2$  of vertices in  $S$  that are joined to one of  $b_1, b_2, \dots, b_5$  (if  $0.7 < |x| < 15$ ) or  $b_1, \dots, b_4$  (if  $|x| < 0.7$ ), and the process  $S_1$  of those that are not. Again let  $E_2$  be the event that no vertices occur in the  $S_2$  process so  $P[E_2] > \exp(-5k)$ . If the events  $F'_2 \cap H_1 \cap E_2$  occur then  $x$  is 2-pivotal. Therefore for any configuration  $\chi$  where  $F'_2$  occurs

$$\begin{aligned} P[E_{n,2}(x)|\chi] & \geq \delta_7 \delta_8 \exp(-5k) \\ & \geq \delta_7 \delta_8 \exp(-5k) P[E_{n,1}|\chi] \\ & =: \varepsilon_4 P[E_{n,1}|\chi] \end{aligned}$$

Case 2;  $F'_2$  does not occur. In this case  $p(x, W) < \mu$ .

For  $x$  to be 1-pivotal one of the following events must occur.

- 1) The vertex  $x$  is joined to  $W$ .
- 2) Some vertex occurs in the red process inside  $Q \setminus B_{0.5}$  which is joined to  $W$ .

The process of red vertices inside  $Q$  that connect to  $W$  is an inhomogeneous Poisson process with intensity  $p\lambda p(z, W)$ . But for any  $z$  inside  $Q$  we have  $|z - x| \leq 23$  so by Proposition 7.3 this means that  $p(x, W) \geq c^{23} p(z, W)$ . Therefore the probability of a vertex occurring in this process is less than the probability of a vertex occurring in a homogeneous process on  $Q$  with intensity  $p\lambda p(x, W) c^{-23}$ . The number of vertices in this process has a Poisson distribution with mean no more than  $K' p(x, W)$  where  $K' := p\lambda c^{-23} \pi (23)^2$ . So

the probability that a vertex occurs is no more than  $K'p(x, W)$  by Markov's inequality. Therefore the probability of one of the two events above happening is no more than  $(1 + K')p(x, W)$  so given a configuration  $\chi$  where  $F_2^c$  occurs the conditional probability of  $x$  being 1-pivotal satisfies:

$$P[E_{n,1}(x)|\chi] \leq (1 + K')p(x, W).$$

If  $0.7 < |x| < 15$  then let the event  $H_1$  be as before in this case. The probability of the vertex  $b_1$  connecting to  $W$  is now at least  $cp(x, W)$  as  $b_1$  is within 1 of  $x$ . Therefore the conditional probability that  $H_1$  occurs is at least

$$\begin{aligned} & [(0.01^2 \pi \lambda p \exp(-0.01^2 \pi \lambda p))^5 \delta_1^4 \kappa(180)^5 \\ & \times f(1)^6 f(16)(1 - f(0.1))^8 \exp(-(23)^2 \pi \lambda)(1 - p)cp(x, W) \quad =: \quad \delta_9 p(x, W). \end{aligned}$$

If  $|x| < 0.7$  then let the event  $H_1$  be as before in this case. The probability of the vertex  $b_3$  connecting to  $W$  is now at least  $cp(x, W)$  as  $b_3$  is within 1 of  $x$ . Therefore the conditional probability that  $H_1$  occurs is at least

$$\begin{aligned} & [(0.01^2 \pi \lambda p \exp(-0.01^2 \pi \lambda p))^4 \delta_1^3 \kappa(180)^4 \\ & \times f(2)^6 (1 - f(0.1))^4 \exp(-(23)^2 \pi \lambda)(1 - p)cp(x, W) \quad =: \quad \delta_{10} p(x, W). \end{aligned}$$

Therefore for any configuration  $\chi$  where  $F_2^c$  occurs we have

$$\begin{aligned} P[E_{n,2}(x)|\chi] & \geq \delta_9 \delta_{10} p(x, W) \exp(-5k) \\ & \geq \delta_9 \delta_{10} \exp(-5k)(1 + K')^{-1} P[E_{n,1}(x)|\chi] \\ & := \varepsilon_5 P[E_{n,1}(x)|\chi]. \end{aligned}$$

Taking  $\gamma_2 := \varepsilon_4 \varepsilon_5$  this completes the proof of Proposition 7.6.  $\square$

**If  $x$  is near the edge of  $B_n$ .** If  $n - 15 \leq |x| \leq n$  then we start off by creating the process of non-red vertices inside  $B_n \setminus C_{23}$  as in Stage 1 but with  $C_{12}$  replaced by  $C_{23}$ . This creates all of the green vertices inside  $B_n \setminus C_{23}$  and builds up the red process on a set  $H$ . Let  $Q = C_{23} \setminus H$ . We again

let  $\mu_1 = c^{17}$ . We let  $T$  be the possibly empty set of red vertices currently connected to  $B_{0.5}$ . We let  $V_0$  be the set of red vertices not in  $T_0$ . Next we build up the set  $T$  of vertices that are connected to  $B_{0.5}$  as in Stages 2 and 3 but this time we stop if we get  $p(x, T) > \mu_1$ . We do not carry out Stages 4 and 5 this time. Define  $F'_1$  to be the event that the algorithm stops with  $p(x, T) > \mu_1$ , so this is the equivalent of  $F_1$ . Then build up the rest of the process outside  $C_{23}$  as in Stage 6, except this time this involves building up the process on  $\overline{B_n} \setminus C_{23}$  as well. Let  $S$  be the vertices that occur in this last stage and let  $E_1$  be the event that there is no current coloured path from  $B_{0.5}$  to  $\overline{B_n}$ . Let  $\chi$  be the configuration of everything that has occurred by the end of Stage 3, i.e. all the vertices except those in  $S$  and whether they are green, red or closed and the edges between them. The following Proposition will complete the proof of Lemma 7.1 for  $x$  near the edge of  $B_n$  and is the equivalent of Proposition 7.5.

**Proposition 7.7** *There exists a function  $\gamma_3(p, q)$  which is strictly positive and continuous on  $(0, 1)^2$  such that for all  $x$  with  $n - 15 \leq |x| \leq n$  and for any possible configuration  $\chi$  the conditional probabilities of  $x$  being 1-pivotal and 2-pivotal satisfy*

$$P[E_{n,2}(x)|\chi] \geq \gamma_3 P[E_{n,1}(x)|\chi].$$

We have 2 cases:

Case 1;  $F'_1$  occurs. Then  $p(x, T) > \mu_1$ . Let  $M$  be the set  $T \setminus (V_0 \cup t')$ . Therefore  $p(x, M) < \mu_1$  and again for any point  $z$  within 16 of  $x$ , we have  $p(z, T) < 1 - \delta_1$ . As before for any point  $y$  within 1 of  $x$  we have  $p(y, W) \geq cp(x, W) > c\mu_1 = \delta_6$ .

Let  $A_1, A_2, A_3, A_4$  be circles of radius 0.01 centred on the points with coordinates  $(\pm 0.3, 0.1)$ ,  $(\pm 0.3, 0.2)$  in relation to  $x$  with the line going from  $x$  to the origin as the  $y$ -axis, so all of these circles are contained in  $B_n$  (see Figure 7.4). Let  $A_5$  be a circle of radius 0.01 centred on a point at distance 15.1 from  $x$  on a line going away from the origin, so  $A_5$  is contained in  $\overline{B_n}$  and any point in  $A_5$  will be within 16 of  $x$  and any point in  $A_3$ .

Let  $H_2$  be the following event:

- 1) We get exactly one red vertex in each of  $A_1, \dots, A_5$ , which we call  $b_1, \dots, b_5$  respectively.
- 2) The vertex  $b_1$  connects to at least one vertex in  $T$  but no vertices outside  $Q$  that are not in  $T$  and  $b_2, b_3, b_4, b_5$  connect to no vertices outside  $Q$ .



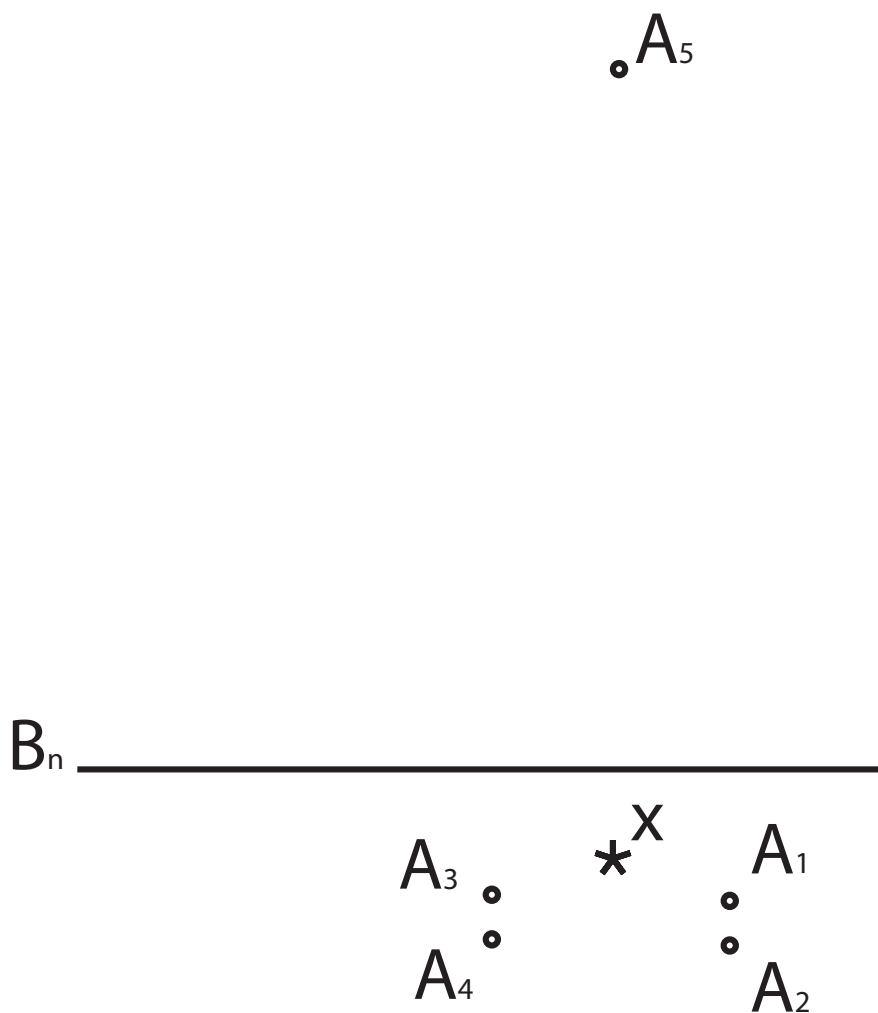


Figure 7.4: For the case  $n - 15 < |x| < n$ .

- 3) There are edges  $b_1 \sim b_2$ ,  $b_3 \sim b_5$  and  $b_3 \sim b_4$  and all of  $b_1, b_2, b_3, b_4$  connect with  $x$  but there are no other edges amongst them.
- 4) There are no other vertices inside  $Q$ .
- 5) The uniform random variable associated with  $x$  satisfies  $Y_0 > p$ .
- The conditional probability that  $H_3$  occurs is at least

$$[(0.01^2 \pi \lambda p \exp(-0.01^2 \pi \lambda p))^5 \delta_1^4 \delta_6 [\kappa(180)]^5 \\ \times f(1)^6 f(16)(1 - f(0.1))^8 \exp(-(23)^2 \pi \lambda)(1 - p) = \delta_7.$$

Now split up Stage 6 of the algorithm into two processes as before, namely the process  $S_2$  of vertices in  $S$  that are joined to one of  $b_1, b_2, \dots, b_5$ , and the process  $S_1$  of those that are not. Again let  $E'_1$  be the event that the vertices in  $S_1$  do not complete a path from  $B_{0.5}$  to  $\overline{B_n}$ . Let  $E_2$  be the event that no vertices occur in the  $S_2$  process. Therefore the events  $E'_1$  and  $E_2$  are independent and  $P[E_2] > \exp(-5k)$ . The intensity of the  $S_1$  process is everywhere less than the intensity of the whole  $S$  process so the probability of  $E'_1$  occurring is greater than the probability of  $E_1$  occurring as they are decreasing events. If the events  $F'_1 \cap H_2 \cap E'_1 \cap E_2$  occur then  $x$  is 2-pivotal. Therefore for any configuration  $\chi$  where  $F'_1$  occurs

$$\begin{aligned} P[E_{n,2}(x)|\chi] &\geq \delta_7 \exp(-5k) P[E_1|\chi] \\ &\geq \delta_7 \exp(-5k) P[E_{n,1}|\chi] \\ &=: \varepsilon_6 P[E_{n,1}|\chi] \end{aligned}$$

Case 2;  $F'_1$  does not occur. In this case  $p(x, T) < \mu$ .

For  $x$  to be pivotal one of the following events must occur.

- 1) The vertex  $x$  is joined to  $T$ .
- 2) Some vertex occurs in the red process inside  $Q \setminus \overline{B_n}$  which is connected to  $T$ .

Again the probability of one of the two events above happening is no more than  $(1 + K')p(x, T)$  so given a configuration  $\chi$  where  $F_1'^c$  occurs the conditional probability of  $x$  being 1-pivotal satisfies;

$$P[E_{n,1}(x)|\chi] \leq (1 + K')p(x, T)P[E_1|\chi].$$

Let the event  $H_2$  be as before. The conditional probability that  $H_2$  occurs is at least

$$\begin{aligned} & [(0.01^2 \pi \lambda p \exp(-0.01^2 \pi \lambda p))^5 \delta_1^4[\kappa(180)]^5 \\ & \times f(1)^6 f(16)(1 - f(0.1))^8 \exp(-(23)^2 \pi \lambda)(1 - p) c p(x, T) = \delta_9 p(x, W). \end{aligned}$$

Therefore for any configuration where  $F_1^c$  occurs we have

$$\begin{aligned} P[E_{n,2}(x)|\chi] & \geq \delta_9 p(x, W) \exp(-5k) P[E_1|\chi] \\ & \geq \delta_9 \exp(-5k)(1 + K')^{-1} P[E_{n,1}(x)|\chi] \\ & =: \varepsilon_7 P[E_{n,1}(x)|\chi]. \end{aligned}$$

Taking  $\gamma_3 := \varepsilon_6 \varepsilon_7$  this completes the proof of Proposition 7.7.  $\square$

Let  $\delta := \min(\gamma_1 \gamma_2 \gamma_3)$  and the proof of Lemma 7.1 is complete.  $\square$

The following proposition follows immediately from Lemmas 7.2 and 7.1.

**Proposition 7.8** *There is a strictly positive continuous function  $\delta : (0, 1)^2 \rightarrow (0, \infty)$  such that for all  $n \geq 100$  and  $(p, q) \in (0, 1)^2$ , we have*

$$\frac{\partial \psi_n(p, q)}{\partial q} \geq \delta(p, q) \frac{\partial \psi_n(p, q)}{\partial p}.$$

**Proof of Theorem 4.3.** Set  $p^* = p_c^{\text{site}}$  and  $q^* = (1/8)(p^*)^2$ . Then using Proposition 7.8 and looking at a small box around  $(p^*, q^*)$ , we can find  $\varepsilon \in (0, \min(p^*/2, 1 - p^*))$  and  $\kappa \in (0, q^*)$  such that for all  $n > 100$  we have

$$\psi_n(p^* + \varepsilon, q^* - \kappa) \leq \psi_n(p^* - \varepsilon, q^* + \kappa).$$

Taking the limit inferior as  $n \rightarrow \infty$ , since  $\psi$  is monotone in  $q$  we get

$$0 < \psi(p^* + \varepsilon, 0) \leq \psi(p^* + \varepsilon, q^* - \kappa) \leq \psi(p^* - \varepsilon, q^* + \kappa).$$

Now set  $p = p^* - \varepsilon$ . Then  $q^* + \kappa \leq p^2$ , so that  $\theta(p, p^2) > 0$ , and by Proposition 7.1, the enhanced model with parameters  $(p, p^2)$  percolates, i.e. has an infinite coloured component, almost surely. As in the end of Section 5 we can now couple the enhanced site process with a bond process in such a way that if the enhanced site process percolates so does the bond process. Therefore, since the enhanced  $(p, p^2)$  site process percolates almost surely, so does the bond process, so  $p_c^{\text{bond}} \leq p < p_c^{\text{site}}$ .  $\square$

## 7.2 Proof of Theorems 4.6 and 4.7

As in the finite range case, we make use of a mixed site and bond percolation model and compare it with a diminished site percolation model in such a way that if the mixed percolation model percolates then so does the diminished site percolation model.

**The diminishment** As in the finite range case label each site an up-site with probability one half and if not then label it a down-site. Label each site red (and open) with probability  $p$  and otherwise it is closed. If a red site  $x$  is a down-site and has exactly one red up-site  $y$  in its 1-neighbourhood which it is connected to but there are no other vertices in its 1-neighbourhood and no downsites in its 2-neighbourhood then we say that  $x$  is correctly configured and we close the edge  $xy$  (which we call a *correctly configured edge*) with probability  $1-q$ , but the sites  $x$  and  $y$  remain red and all other edges involving  $x$  remain open. (see diagram).

We build this model by having a Poisson process of intensity  $\lambda$  and labelling vertices  $x_1, x_2, \dots$  in order of distance from the origin. We also have independent uniform random variables  $W_i, Y_i, Z_i$  for  $i = 0, 1, 2, \dots$ . We say vertex  $x_i$  is an up-site if and only if  $W_i < 1/2$ . A vertex  $x_i$  is red if and only if  $Y_i < p$ . If a vertex  $x_i$  is correctly configured then the edge to its 1-neighbour remains open if and only if  $Z_i < q$ .

Let  $A_n$  be the event that there is an open path from  $B_{0.5}$  to  $\overline{B_n}$  where all vertices outside  $B_n$  are fixed to be red and up-sites. Define  $A_n^x$  similarly in terms of the process with an added vertex at  $x$ .

The set of open edges may be viewed as a *diminishment* of the original set of open edges, in which each correctly configured edge is removed with probability  $1-q$ . We can couple the diminished site percolation process to the mixed bond-site process (with parameters  $p, q$ ) in such a way that if the mixed process percolates then so does the diminished site process, as follows.

Determine the open sites and edges for the mixed bond-site process. Deem each vertex to be red if and only if it is open in the mixed process. If an edge  $xy$  is correctly configured, then close it if and only if it is closed in the mixed process. In this way the set of open sites is exactly the same in each process and if an edge is open in the mixed process then it is open in the diminished site process.

Therefore if there is an infinite open path in the mixed percolation process, it will also be an infinite open path in the diminished site process.

Let  $\theta_n(p, q)$  be the probability that  $A_n$  occurs and let  $\theta(p, q)$  be the limit

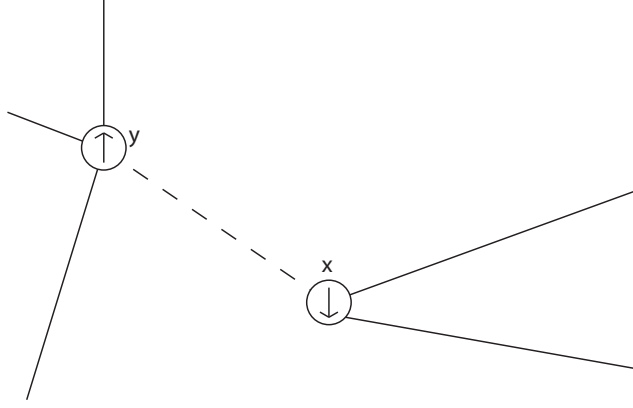


Figure 7.5: Here the vertex  $x$  is correctly configured but only the edge  $xy$  can be removed.

inferior. The proof of Proposition 7.1 is easily modified to this model to give the following result.

**Proposition 7.9** *There is a.s. an infinite connected component using only coloured vertices in the original process if and only if  $\theta(p, q) > 0$*

We say that a point  $x$  is 3-pivotal if putting a vertex at  $x$  and making  $Y_0 < p$  means that  $A_n^x$  occurs but having  $Y_0 > p$  means it does not. Similarly with 4-pivotal and  $Z_0$ . Again we have a form of the Margulis-Russo formulae:

$$\frac{\partial \theta_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,3}(x, p, q) \, dx \quad (7.8)$$

and

$$\frac{\partial \theta_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,4}(x, p, q) \, dx. \quad (7.9)$$

We then need to prove the equivalent of Lemma 5.2:

**Lemma 7.2** *Suppose the random connection function  $f$  is non-increasing and satisfies conditions (7.1), (7.2), (7.3), then there exists a function  $\delta(p, q)$  such that for all  $x \in B_n$ ,*

$$P_{n,4}(x, p, q) \geq \delta(p, q) P_{n,3}(x, p, q)$$

where  $\delta$  is independent of  $n$  and  $x$ , and is strictly positive and continuous on  $(0, 1)^2$ .

**Proof of Lemma 7.2.**

Most of the proof is along the lines of the proof of Lemma 7.1. Again choose  $c \in (0, 0.1)$  such that  $f(r + 1) \geq cf(r)$  for all  $r > 0$  and let  $k = \int_0^\infty 2\pi\lambda r f(r) dr$ , which is the expected degree of a vertex in  $C$ .

Fix  $x$  and again write  $C_r$  for  $C_r(x)$ . Again for any set of vertices  $S$  and any point  $z$  let  $p(z, S)$  be the probability that a vertex at  $z$  is joined to at least one of the vertices in  $S$ . So  $1 - p(z, S) = \prod_{s \in S} (1 - f(|z - s|))$ . Let  $\gamma = \min\{c^2, f(13)\}$ .

**If  $x$  is not near the origin or the edge of  $B_n$**  For now we consider  $x$  with  $15 < |x| < n - 15$ . We then build up the process outside  $C_{12}$  in stages as follows.

**Stage 1.** We start off by creating the process of closed vertices inside  $B_n \setminus C_{10}$ , which is a homogeneous Poisson Process with intensity  $\lambda(1 - p)$ , and decide which are upsites and which are downsites.

**Stage 2.** Next build up the set of possibly correctly configured sites as follows. Let  $U'_i$  be some ordering of squares of the form  $[l, l + 1] \times [m, m + 1]$  where  $l$  and  $m$  are integers and the square intersects  $B_n \setminus C_{12}$ . Then let  $U_i$  be the intersection of  $U'_i$  with  $B_n \setminus C_{12}$ . In each  $U_i$  build up the process of red downsites downwards (a process with intensity  $0.5\lambda p$ ) either until two red downsites occur or until the process has been built up on the whole of  $U_i$ . Let  $L$  be the set of red downsites so far. Add in edges with the appropriate probability. As each  $U_i$  has diameter less than 2, if more than one downsite occurs neither can end up being correctly configured. If a vertex  $x_i$  is the only red downsite in  $U_i$  and there are no downsites currently within 2 of  $x_i$  or upsites currently within 1 of  $x_i$  then  $x_i$  could possibly become correctly configured, and we put it in a set  $G$ . Let  $H_1$  be the region where the process of red downsites has been built up so far.

Then for each vertex  $y$  in  $G$  we build up the rest of the process of red downsites radially outwards in its 2-neighbourhood (a process of intensity  $0.5\lambda p \mathbf{1}_{z \in (C_2(y) \cap B_n) \setminus H_1}$ ) until a vertex occurs or until the whole neighbourhood has been covered. Add in edges appropriately and any extra region where the process of red downsites has been built in  $C_2(y)$  is added to  $H_1$  before moving on to the next vertex in  $G$ . Let  $G'$  be the set of vertices in  $G$  which

now have no red downsites in their 2-neighbourhood. Let  $H_2$  be the area where the red downsite process has now been built up.

For each vertex  $y$  in  $G'$  we now build up the process of red upsites radially outwards in its 1-neighbourhood (a process of intensity  $0.5\lambda p$  inside  $B_n$  and  $\lambda$  outside  $B_n$ ) until two vertices have occurred or until the whole neighbourhood has been covered. Add in edges appropriately. Let  $H_3$  be the area where the red upsite process has been built up and if a vertex in  $G'$  has exactly one red upsite in its 1-neighbourhood and there is an edge to it then it is correctly configured and the edge is removed with probability  $1 - q$ . This is the complete set of correctly configured sites outside  $C_{12}$ .

**Stage 3.** In this stage we build up our starting set of red vertices that are connected to  $B_{0.5}$ . Let  $\mu(z) = 0.5[\mathbf{1}_{z \notin H_2} + \mathbf{1}_{z \notin H_3}]$ . We build up the rest of the process of red vertices outwards from the origin in  $B_{0.5}$ , adding vertices to  $T$ . This is a process with intensity  $p\lambda\mu(z)$  as we have already built up the red downsites in  $H_2$  and the red upsites in  $H_3$ . Again we add in edges as we go and decide if a vertex is an upsite or a downsite appropriately (but it does not matter for the rest of the proof whether they are upsites or downsites, all that matters is that they are red), and we stop if  $p(x, T) > \gamma$  (in which case we go to Step 6 of Stage 4 below) or if we have built up the whole process in  $B_{0.5}$ . We then let  $T_0$  be the set of red vertices connected via a coloured path to  $B_{0.5}$  (this is the same as the set  $T$  for now but  $T_0$  stays fixed, whilst we add vertices to  $T$  at the next stage). We let  $W$  be the possibly empty set of red vertices currently connected to  $\overline{B_n}$ . We let  $V_0$  be the set of red vertices not in either  $T_0$  or  $W_0$ . In any square of side 1 there are at most 8 vertices in  $L$ . So in a square of side 8 there are at most 512 vertices in  $L$ . Each of these corresponds to at most 3 vertices of  $V_0$  by the construction in Stage 2 so in any region of diameter 2 there are at most 1536 vertices in  $V_0$ .

**Stage 4.** The next stage is to build up the set  $T$  generation by generation from our starting point of  $T_0$ , stopping if  $p(x, T)$  becomes high enough. We add in vertices one by one that are connected to the latest generation of  $T$ , stopping if the probability of  $x$  being joined to  $T$  becomes high enough. As we add in vertices one by one, this probability can never jump by too much so we also have an upper bound on the probability of  $x$  being joined to  $T$ . The algorithm goes as follows;

Step 1) Let  $j = 0$

Step 2) Let  $k = 0$  and  $|y_0| = 0.5$

Step 3) We build radially symmetrically outwards from  $B_{|y_k|}$  with intensity  $p\lambda\mu(z)p(z, T_j)[1 - p(z, T \setminus T_j)]\mathbf{1}_{z \in B_n \setminus (B_{0.5} \cup C_{12})}dz$  either until we get a vertex  $y_{k+1}$  or until we have exhausted  $B_n$ . If we do not get a vertex then we go to Step 5.

Step 4) We add  $y_{k+1}$  to  $T_{j+1}$  and add in all edges involving  $y_{k+1}$  conditional on there being at least one edge to the set  $T_j$  and no edges to the set  $T \setminus T_j$ . Any vertices in  $V_j$  that are now connected via a coloured path to  $B_{0.5}$  we add to  $T_{j+1}$ . If we now have  $p(x, T \cup T_{j+1}) > \gamma$  then we go to Step 6, if not we increase  $k$  by one and repeat Step 3.

Step 5) If  $T_{j+1} = \emptyset$  we go to Step 7. If not then we add  $T_{j+1}$  to  $T$  and we let  $V_{j+1} = V_j \setminus T_{j+1}$ . We increase  $j$  by 1 and go back to Step 2.

Step 6) We say the event  $F_1$  has occurred. We add  $T_{j+1}$  to  $T$ , we let  $t'$  be  $y_{k+1}$  and we stop.

Step 7) The event  $F_1$  has not occurred.

**Stage 5.** We then build up the set  $W$  of vertices that are connected to  $\overline{B_n}$  in similar fashion. This stage is the analogue of Stage 3 as we build up the starting point for  $W$ . We build outwards from  $B_n$  with intensity  $\lambda$ , stopping if we get  $p(x, W) > \gamma$  (in which case we then go to Step 6 of Stage 6 below). We then let  $W_0$  be the set of vertices connected by a coloured path to  $\overline{B_n}$ . We let  $Y_0 = V_0 \setminus (W_0 \cup T)$ .

**Stage 6.** This is the analogue of Stage 4 as we build up  $W$  generation by generation. The algorithm is as follows.

Step 1) Let  $j = 0$ ,

Step 2) Let  $k = 0$  and  $|x_0| = 0.5$

Step 3) We build radially symmetrically outwards from  $B_{|x_k|}$  with intensity  $p\lambda\mu(z)p(z, W_j)[1 - p(z, W \setminus W_j)][1 - p(z, T)]\mathbf{1}_{z \in B_n \setminus (B_{0.5} \cup C_{12})}dz$  either until we get a vertex  $x_{k+1}$  or until we have exhausted  $B_n$ . If we do not get a vertex then we go to Step 5.

Step 4) We add  $x_{k+1}$  to  $W_{j+1}$  and add in edges to  $x_{k+1}$  appropriately. Any vertices in  $Y_j$  that are now connected via a coloured path to  $\overline{B_n}$  we add to  $W_{j+1}$ . If we now have  $p(x, (W \cup W_{j+1})) > \gamma$  then we go to Step 6, if not we increase  $k$  by one and repeat Step 3.

Step 5) If  $W_{j+1} = \emptyset$  we go to Step 7. If not then we add  $W_{j+1}$  to  $W$  and we let  $Y_{j+1} = Y_j \setminus W_{j+1}$ . We increase  $j$  by 1 and go back to Step 2.

Step 6) We say the event  $F_2$  has occurred, we add  $W_{j+1}$  to  $W$ , and let  $w'$  be  $x_{k+1}$  and we stop.



Step 7) The event  $F_2$  has not occurred.

**Stage 7.** We now build up the rest of the process outside  $C_{12}$  in similar fashion to Stage 6 of Section 7.1.

We then add in all edges involving these new vertices, conditional on having no edges to  $T$  and  $W$  that should not be there. We may also have to finish building the process of red vertices inside  $B_{0.5}$  or outside  $B_n$ . We let  $E_1$  be the event that there is no current path from  $B_{0.5}$  to  $\overline{B_n}$ , so this must occur for  $x$  to be pivotal. We let  $Q$  be the region we have not built up the process in yet, so  $Q$  is contained in  $C_{12}$ . We let  $V'$  be the set of red vertices that were in  $V_0$  and are not in  $T$  or  $W$ . We let  $S$  be the set of red vertices not in  $T$ ,  $V'$  or  $W$ , which are the vertices that occurred in Stage 7.

Let  $\chi$  be the configuration of everything that has occurred by the end of Stage 6, i.e. all the vertices except those in  $S$  and whether they are red upsites or red downsites or closed upsites or downsites and the edges between them. Now the following Proposition will complete the proof of Lemma 7.2 for  $x$  not near  $B_{0.5}$  or the edge of  $B_n$ .

**Proposition 7.10** *There exists a function  $\eta_1(p, q)$  which is strictly positive and continuous on  $(0, 1)^2$  such that for all  $x$  with  $15 < |x| < n - 15$  and for any possible configuration  $\chi$  the conditional probabilities of  $x$  being 3-pivotal and 4-pivotal satisfy:*

$$P[E_{n,4}(x)|\chi] \geq \eta_1 P[E_{n,3}(x)|\chi].$$

**Proof** We have 3 cases:

Case 1; Both  $F_1$  and  $F_2$  occur. We have that  $p(x, T) > \gamma$  and  $p(x, W) > \gamma$ . We also have  $p(x, T \setminus (V_0 \cup t')) < \gamma$  (as otherwise the algorithm would have stopped earlier).

We let  $A$  be a circle of radius 0.1 centred on the point with coordinates  $(0.5, 0.5)$  in relation to  $x$ . Now for any point  $y$  within 1 of  $x$  and taking  $M$  to be the set  $T \setminus (V_0 \cup t')$  we can apply Proposition 7.3 to get  $p(y, M) \leq c^{-1}p(x, M) \leq c^{-1}\gamma < 0.1$ . Also  $p(y, t') < 0.1$  and by applying Proposition 7.4 to  $V_0$  we get  $p(y, V_0) < 1 - \kappa(1536)$ . Therefore  $1 - p(y, T) > 0.9^2\kappa(1536) =: \delta_1$  for any  $y$  in  $C_1$ . Similarly  $1 - p(y, W) > \delta_1$  for any  $y$  in  $C_1$ .

Taking  $M$  to be the set  $T$  we have  $p(y, T) \geq cp(x, T) > c\gamma := \delta_2$ . Similarly  $p(y, W) > \delta_2$ .

Therefore if we have exactly one red vertex in  $A$  it will be connected to  $T$  but not  $W$  or  $V'$  with probability at least  $\delta_1\delta_2\kappa(1536)$ . Similarly the

probability that a vertex at  $x$  is connected to  $W$  but not  $T$  or  $V'$  is at least  $\delta_1\delta_2\kappa(1536)$ .

Let  $G$  be the following event:

- 1) There is exactly one red vertex in  $A$ , which we call  $b$ .
- 2) The vertex  $b$  connects to at least one vertex in  $T$  but no vertices outside  $Q$  that are not in  $T$ .
- 3) The vertex  $x$  connects to at least one vertex in  $W$  but no vertices outside  $Q$  that are not in  $W$ .
- 3) There is an edge between  $b$  and  $x$ .
- 4) There are no other vertices inside  $Q$ .
- 5) The uniform random variable associated with  $x$  satisfies  $Y_0 > p$ .
- 6) The vertex  $x$  is a downsite and  $b$  is an upsite.

The conditional probability that  $G$  occurs is at least

$$\begin{aligned} \frac{1}{4}[(0.1^2\pi\lambda p \exp(-0.1^2\pi\lambda p)]\delta_1^2\delta_2^2[\kappa(1536)]^2 \\ \times f(1) \exp(-(12)^2\pi\lambda)(1-p) =: \delta_3. \end{aligned}$$

We now split up Stage 7 of the algorithm into two processes. The process  $S_2$  of vertices in  $S$  that are joined to one of  $x, b$  and the process  $S_1$  of those that are not. The number of vertices in  $S_2$  has a Poisson distribution with mean less than  $2k$  as  $k$  is the mean number of neighbours of a vertex in an infinite process. Let  $E'_1$  be the event that the vertices in  $S_1$  do not complete a path from  $B_{0.5}$  to  $\overline{B}_n$ . Let  $E_2$  be the event that no vertices occur in the  $S_2$  process. Therefore the events  $E'_1$  and  $E_2$  are independent and  $P[E_2] > \exp(-2k)$ . The intensity of the  $S_1$  process is everywhere less than the intensity of the whole  $S$  process so the probability of  $E'_1$  occurring is greater than the probability of  $E_1$  occurring as they are decreasing events.

If the events  $F_1 \cap F_2 \cap G \cap E'_1 \cap E_2$  occur then  $x$  is 4-pivotal. Therefore for any configuration  $\chi$  where  $F_1 \cap F_2$  occurs

$$\begin{aligned} P[E_{n,4}(x)|\chi] &\geq \delta_3 \exp(-2k)P[E_1|\chi] \\ &\geq \delta_3 \exp(-2k)P[E_{n,3}|\chi] \\ &=: \varepsilon_1 P[E_{n,3}|\chi] \end{aligned}$$

Case 2; Both  $F_1$  and  $F_2$  do not occur. This means that  $p(x, T) < \gamma$  and  $p(x, W) < \gamma$ . For  $x$  to be 3-pivotal one of the following events must occur.

- 1) The vertex  $x$  is joined to  $T$  and  $W$ .
- 2) The vertex  $x$  is joined to  $T$  and some vertex occurs in the red process inside  $Q$  that connects to  $W$ .
- 3) The vertex  $x$  is joined to  $W$  and some vertex occurs in the red process inside  $Q$  that connects to  $T$ .
- 4) Some vertex occurs in the red process inside  $Q$  that connects to  $W$  and another vertex occurs that connects to  $T$ .

The process of red vertices inside  $Q$  that connect to  $T$  is an inhomogeneous Poisson process with intensity  $p\lambda\mu(z)p(z, T)$  (where  $\mu(z)$  is as defined in Stage 3). But for any  $z$  inside  $Q$  we have  $|z - x| \leq 12$  so by Proposition 7.3 this means that  $p(x, T) \geq c^{12}p(z, T)$ . Therefore the probability of a vertex occurring in this process is less than the probability of a vertex occurring in a homogeneous process on  $Q$  with intensity  $p\lambda p(x, T)c^{-12}$ . The number of vertices in this process has a Poisson distribution with mean no more than  $Kp(x, T)$  where  $K := p\lambda c^{-12}\pi(12)^2$ . So the probability that a vertex occurs is no more than  $Kp(x, T)$  by Markov's inequality. Therefore the probability of one of the events above happening is no more than  $(1 + K)^2p(x, T)p(x, W)$  so given a configuration  $\chi$  where  $F_1^c \cap F_2^c$  occurs the conditional probability of  $x$  being 1-pivotal satisfies:

$$P[E_{n,3}(x)|\chi] \leq (1 + K)^2p(x, T)p(x, W)P[E_1|\chi].$$

Again let  $A$  be a circle of radius 0.1 centred on the point with coordinates  $(0.5, 0.5)$  in relation to  $x$ . Now for any point  $y$  within 1 of  $x$  again we have  $p(y, T) \geq cp(x, T)$  and we have  $1 - p(y, W) \geq \delta_1$ .

Therefore if we have exactly one red vertex in  $A$  it will be connected to  $T$  but not  $W$  or  $V'$  with probability at least  $\delta_1\kappa(1536)cp(x, T)$ .

Similarly the probability that a vertex at  $x$  is connected to  $W$  but not  $T$  or  $V'$  is at least  $\delta_1\kappa(1536)p(x, W)$ .

Let  $G$  be the same event as before. The conditional probability that  $G$  occurs is at least

$$\begin{aligned} \frac{1}{4}[(0.1^2\pi\lambda p \exp(-0.1^2\pi\lambda p))\delta_1^2[\kappa(1536)]^2cp(x, T)p(x, W) \\ \times f(1) \exp(-(12)^2\pi\lambda)(1 - p) \quad =: \quad \delta_4p(x, T)p(x, W). \end{aligned}$$

Now we carry out Stage 7 of the algorithm and let  $E'_1$  and  $E_2$  be as before. Again  $P[E'_1|\chi] \geq P[E_1|\chi]$  and  $P[E_2] > \exp(-2k)$ . If the events

$F_1^c \cap F_2^c \cap G \cap E_1' \cap E_2$  occur then  $x$  is 4-pivotal. Therefore for any configuration  $\chi$  where  $F_1^c \cap F_2^c$  occurs

$$\begin{aligned} P[E_{n,4}(x)|\chi] &\geq \delta_4 p(x, T) p(x, W) \exp(-2k) P[E_1|\chi] \\ &\geq \frac{\delta_4 \exp(-2k)}{(1+K)^2} P[E_{n,3}|\chi] \\ &=: \varepsilon_2 P[E_{n,3}|\chi]. \end{aligned}$$

Case 3; The event  $F_1$  occurs but  $F_2$  does not. This means that for  $x$  to be 3-pivotal one of the following events must occur.

- 1) The vertex  $x$  is joined to  $W$ .
- 2) Some vertex occurs in the red process inside  $Q$  that connects to  $W$  but not  $T$ .

By the same argument as for Case 2 the probability of one of the two events above happening is no more than  $(1+K)p(x, W)$  so given a configuration  $\chi$  where  $F_1 \cap F_2^c$  occurs the conditional probability of  $x$  being 3-pivotal satisfies;

$$P[E_{n,3}(x)|\chi] \leq (1+K)p(x, W)P[E_1|\chi].$$

Let  $G$  be the same event as before. By similar arguments to the previous 2 cases the conditional probability that  $G$  occurs is at least

$$\begin{aligned} &\frac{1}{4}[(0.1^2 \pi \lambda p \exp(-0.1^2 \pi \lambda p)) \delta_1^2 [\kappa(1536)]^2 \delta_2 p(x, W) \\ &\quad \times f(1) \exp(-(12)^2 \pi \lambda)(1-p) =: \delta_5 p(x, W). \end{aligned}$$

Now we carry out Stage 7 of the algorithm and let  $E_1'$  and  $E_2$  be as before. Again  $P[E_1'|\chi] \geq P[E_1|\chi]$  and  $P[E_2] > \exp(-2k)$ . If the events  $F_1 \cap F_2^c \cap G \cap E_1' \cap E_2$  occur then  $x$  is 4-pivotal. Therefore for any configuration  $\chi$  where  $F_1 \cap F_2^c$  occurs

$$\begin{aligned} P[E_{n,4}(x)|\chi] &\geq \delta_5 p(x, W) \exp(-2k) P[E_1|\chi] \\ &\geq \frac{\delta_5 \exp(-2k)}{1+K} P[E_{n,1}|\chi] \\ &=: \varepsilon_3 P[E_{n,1}|\chi]. \end{aligned}$$

Case 4; The event  $F_2$  occurs but  $F_1$  does not. The same arguments as Case 3 but with the roles of  $T$  and  $W$  reversed leads to the following equation for these configurations,

$$P[E_{n,2}(x)|\chi] \geq \varepsilon_3 P[E_{n,1}|\chi].$$

Therefore if we take  $\eta_1 := \varepsilon_1 \varepsilon_2 \varepsilon_3$  the proof of Proposition 7.10 is complete.

□

This shows that if  $x$  is not near  $B_{0.5}$  or the edge of  $B_n$  then

$$P_{n,4}(x) > \eta_1(p, q) P_{n,3}(x).$$

If  $x$  is near the origin or the edge of  $B_n$  we can deal with it as in the previous section to get some  $\eta_2$  such that:

$$P_{n,4}(x) > \eta_2(p, q) P_{n,3}(x).$$

Let  $\delta := \eta_1 \eta_2$  and the proof of Lemma 7.2 is complete. □

**Proof of Theorem 4.6.**

Lemma 7.2 and equations (7.8) and (7.9) immediately show that

$$\frac{\partial \theta_n(p, q)}{\partial q} \geq \delta(p, q) \frac{\partial \theta_n(p, q)}{\partial p}. \quad (7.10)$$

We take  $q_0 < 1$ , and fix  $f$ . We now set  $q^* = (1 + q_0)/2$ , and choose  $\lambda > \lambda_{q_0 f}$ , and consider the graph  $RCM(\lambda, f)$ . Define  $p^* := \lambda_{q_0 f}/\lambda$ , so  $p^* \in (0, 1)$ . Now by considering a small box around  $(p^*, q^*)$ , and using equation (7.10) above we can find  $\varepsilon > 0$  such that  $\varepsilon < \min(p^*, 1 - p^*)$  and

$$\theta(p^* + \varepsilon, q_0) \leq \theta(p^*, q^*) \leq \theta(p^* - \varepsilon, 1).$$

Now the definition of  $p^*$  together with Proposition 7.9 implies that  $RCM((p^* + \varepsilon)\lambda, q_0 f)$  percolates. Hence, on  $RCM(\lambda, f)$  the mixed site-bond process with parameters  $(p^* + \varepsilon, q_0)$  percolates and therefore the diminished site process with parameters  $(p^* + \varepsilon, q_0)$  percolates. Thus  $\theta(p^* + \varepsilon, q_0) > 0$  and hence  $\theta(p^* - \varepsilon, 1) > 0$  which means that  $RCM((p^* - \varepsilon)\lambda, f)$  percolates so

$$\lambda_f \leq (p^* - \varepsilon)\lambda < p^*\lambda = \lambda_{q_0 f}$$

and we are done. □

**Proof of Theorem 4.7.** Since  $S_p f \equiv S_{p/q} S_q f$ , it suffices to consider the case with  $q = 1$  so that  $S_q f \equiv f$ . Define the connection function  $g(r) = f(\sqrt{p}r)$ . Then  $S_p f \equiv pg$  so by Theorem 4.6,

$$\lambda_{S_p f} > \lambda_g$$

and hence the graph  $\text{RCM}(\lambda_{S_p f}, g)$  is the realization of a supercritical random connection model. Let  $p_c^{\text{bond}}$  and  $p_c^{\text{site}}$  denote the critical values for bond, respectively site, percolation on this graph. By Theorem 4.3, we have  $p_c^{\text{bond}} > p_c^{\text{site}}$ .

Given  $p' \in (0, p)$ , we have  $p'g \equiv (p'/p)S_p f$  so that  $\lambda_{S_p f} < \lambda_{p'g}$  by Theorem 4.6, so that the graph  $\text{RCM}(\lambda_{S_p f}, p'g)$  does not percolate, and therefore  $p_c^{\text{bond}} \geq p'$ . Hence,

$$p \leq p_c^{\text{bond}} < p_c^{\text{site}}. \quad (7.11)$$

By scaling, the graph  $\text{RCM}(\lambda_{S_p f}, g)$  is equivalent to the graph  $\text{RCM}(p^{-1}\lambda_{S_p f}, f)$  so that

$$p_c^{\text{site}} = \frac{p\lambda_f}{\lambda_{S_p f}}$$

and combining this with (7.11) yields the desired inequality  $\lambda_{S_p f} < \lambda_f$ .

For the last part, observe that  $\int_0^\infty r S_p f(r) dr = \int_0^\infty r f(r) dr$  and therefore the fact that (2.3) holds as a weak inequality for  $S_p f$  implies that it holds as a strict inequality for  $f$ .  $\square$

## 8 Random Sequential Adsorption

This chapter introduces the random sequential adsorption model on the square lattice which is the subject of the second half of the thesis. Colour the sites of the lattice in a chequerboard pattern so we define  $(0, 0)$  to be blue and then  $(1, 0), (-1, 0), (0, 1), (0, -1)$  to be red and so on, so a red site is always adjacent to four blue sites and vice versa. Then assign independent Poisson arrival process at each site with rate 1 on the red sites and rate  $\lambda$  on the blue sites, starting off with all sites empty. Let  $t_x$  be the time of the first arrival in the Poisson process at  $x$ . Then if none of the four neighbours of  $x$  are occupied at  $t_x$  define  $x$  to be occupied from then on. If one or more of its neighbours are occupied then  $x$  becomes blocked, and remains so from then on. In this way every site will eventually end up being occupied or blocked (see Penrose and Sudbury [21]), this is the *jammed state*. A *stable set* is a set of vertices where no two vertices are adjacent, so the set of occupied sites is a maximal stable set.

If a blue site is occupied we declare it to be black and if it is blocked we declare it to be white. If a red site is occupied we declare it to be white and if it is blocked we declare it to be black. We can then form a graph of black vertices with edges between any two black vertices that are adjacent in the square lattice. Let  $\theta(\lambda)$  be the probability that there is an infinite black component containing the origin. Then by a coupling argument it can be shown that  $\theta(\lambda)$  is monotonically increasing in  $\lambda$ . By this fact and by Kolmogorov's 0 – 1 law there is a critical value  $\lambda_c$  above which there will almost surely be an infinite black component and below which there will almost surely not be an infinite black component.

### 8.1 Known results and applications

This model is of interest as it is a dependent percolation model, about which not a lot is known. In this thesis the result that  $\lambda_c > 1$  is proved and it provides another use of the enhancement technique. Random sequential adsorption is of interest in physics, for example in coating a surface with a substance. The percolation element of the model could be used to model the electrical conductivity of the coating for example. Penrose and Sudbury [21] show that if  $\lambda = 1$  the probability a site ends up occupied is at least  $1/3$ . Simulations suggest that the probability a site ends up occupied is about 0.364 with  $\lambda = 1$  (see page 1292 of Evans [5]). In the independent

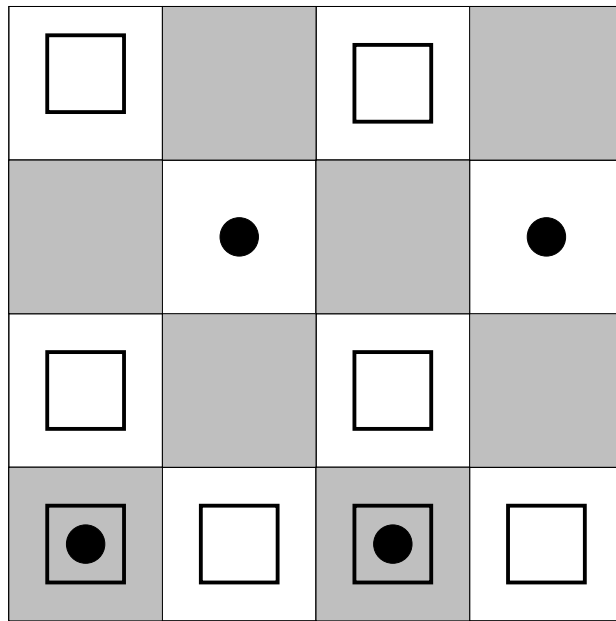


Figure 8.1: Example: The shaded squares are blue sites and the white squares are red sites, the squares with a circle in are open sites, the squares with a black square in are black sites and the squares without a black square in are white sites



site percolation model on  $\mathbb{Z}^2$  we have that the critical probability  $p_c^{site}$  is strictly greater than 0.5 (see Wierman [25]) so the result that  $\lambda_c > 1$  is an analogue of that result in this dependent percolation model. An important result in dependent percolation was that the critical probability for Voronoi percolation is equal to  $1/2$ . This was proved by Bollobas and Riordan [3] and included an RSW (Russo-Seymour-Welsh) type result for dependent percolation. Although they proved the result for Voronoi percolation it can be used for other dependent percolation models, for example in the contact process [2]. Here we use it in the random sequential adsorption model.

## 8.2 Harris-FKG inequality

Given a set of independent variables  $X_1, X_2, \dots$  which take values in the closed interval  $[0, 1]$  we say an event  $A$  on these variables is an *increasing event* if for any set  $X'_i$  where  $A$  occurs, making any variable bigger means that  $A$  still occurs. The Harris-FKG inequality says that for any increasing events  $A$  and  $B$  we have the following positive correlation:

$$P[A \cap B] \geq P[A]P[B]$$

We can also have a sense of an event being increasing on the RSA model described above. Looking purely at the  $\mathbb{Z}^2$  lattice with sites being black or white an event can be increasing if making a site black cannot stop the event from happening. Penrose and Sudbury [21] proved in their paper that the Harris-FKG inequality holds here too. They did this by having arrival times uniformly distributed on  $[0, 1]$  instead of exponential, and then showed that making the arrival later on a red square or earlier on a blue square could never stop an increasing event from happening. In this way an increasing event on the RSA model is the equivalent of an increasing event on a  $[0, 1]^\infty$  model where we have arrival times for the red sites and 1 minus the arrival times for the blue sites.

## 8.3 Result

The main result for the RSA model are the following bounds for  $\lambda_c$ .

**Theorem 8.1**  $1 < \lambda_c < 10$

The upper bound is easy to prove and using results from [3] the non-strict lower bound is easy to prove so most of the work involved in the proof of this is in using the enhancement technique to prove the strict inequality in the lower bound.

## 9 Random Sequential Adsorption: Proof of Theorem 8.1

### 9.1 Proof of the upper bound

The upper bound,  $\lambda_c < 10$  is simple to prove, and is dealt with first. Start by colouring all sites yellow that have even coordinates adding up to a multiple of 4, such as  $(0, 0)$ ,  $(2, 2)$ ,  $(0, 4)$  and so on. Define a square lattice of yellow sites by saying two yellow sites are adjacent if they are  $2\sqrt{2}$  apart. Then consider site percolation on this lattice with each site being occupied independently with probability  $\frac{\lambda}{4+\lambda}$ . This corresponds to the probability that an arrival at a yellow site happens before an arrival at any of its neighbours in the original lattice. If two adjacent yellow sites are occupied in the new lattice then they are occupied in the original lattice and also the even site midway between them will also be occupied. Therefore if there is an infinite component in the new lattice there is also one in the original lattice, so we have the following inequality:

$$\frac{\lambda_c}{4 + \lambda_c} \leq p_s$$

where  $p_s$  is the critical site probability on the square lattice, which is known to be less than 0.7 (Wierman [25]). Rearranging gives that

$$\lambda_c \leq \frac{4p_s}{1 - p_s} < \frac{28}{3} < 10$$

so this proves the upper bound.  $\square$

In the remaining sections, the lower bound  $\lambda_c > 1$  is proved. Although the result is perhaps to be expected by analogy with known (though non-trivial) results for Bernoulli (i.e., independent) site percolation, we are not aware of any such results in a dependent site percolation setting such as we consider here. By use of the weak RSW-type lemma established by Bollobás and Riordan [3] for percolative systems enjoying weak dependence, we shall rather quickly establish the weak version of the inequality, namely  $\lambda_c \geq 1$  (see Remark 9.1). To make this inequality strict we use the technique of enhancement. While this technique is well known, in the present setting its application is quite intricate, requiring a whole sequence of notions of pivotal vertex (see Sections 9.3 and 9.4).

## 9.2 Duality

Define the dual lattice  $\Lambda^*$  to be the square lattice  $\Lambda$  with the diagonals added so that two sites are adjacent if their centres are at distance 1 or  $\sqrt{2}$  from each other. On any square set of sites we have exactly one of the following two events, either a black horizontal crossing in  $\Lambda$  or a white vertical crossing in  $\Lambda^*$ .

Define  $f_\lambda(\rho, s)$  to be the probability that there is a horizontal black crossing of the rectangle  $[1, 2\lfloor \frac{\rho s}{2} \rfloor] \times [1, 2\lfloor \frac{s}{2} \rfloor]$  (an approximately  $\rho s \times s$  lattice rectangle with even side lengths). Define  $f_\lambda^*(\rho, s)$  to be the probability that there is a horizontal black crossing of this rectangle when we allow diagonal edges as well.

In subsequent sections, we shall prove the following key result.

**Proposition 9.1** *There exists  $\mu < 1$  such that*

$$\liminf_{s \rightarrow \infty} f_\mu^*(1, s) > 0. \quad (9.1)$$

In the remainder of the present section, we show how to complete the proof of Theorem 8.1, given Proposition 9.1. The argument uses two further results, which we give now.

We say site  $x \in \mathbb{Z}^2$  *affects* site  $y \in \mathbb{Z}^2$  if there exists a self-avoiding path in  $\mathbb{Z}^2$  starting at a neighbour of  $x$  and ending at  $y$ , with arrival times occurring in increasing order along this path. If  $x$  does not affect  $y$ , then any change to  $t_x$  (with other arrival times unchanged) will not cause any change to the occupied/blocked status of site  $y$ . Similarly to arguments in [17], we have the following simple lemma.

**Lemma 9.1** *Let  $x \in \mathbb{Z}^2$ . The probability that site  $x$  is affected from distance greater than  $r$  tends to zero faster than exponentially as  $r \rightarrow \infty$ . Likewise, the probability that site  $x$  affects some site at distance greater than  $r$  from  $x$  tends to zero faster than exponentially as  $r \rightarrow \infty$ .*

**Proof.** For any self-avoiding path of length  $r$ , taking alternate sites along the path one has at least  $\lfloor r/2 \rfloor$  independent identically distributed arrival times, so the probability they occur in increasing order is at most  $1/\lfloor r/2 \rfloor!$ . Therefore the probability that  $x$  is affected from distance greater than  $r$  is at most  $4(3^r)/\lfloor r/2 \rfloor!$ , which tends to zero faster than exponentially as  $r \rightarrow \infty$ . The proof of the second part is similar.  $\square$

We also use the following much deeper lemma, which is a weak version of the RSW lemma for dependent percolation.

**Proposition 9.2** *Let  $\lambda > 0$  and  $\rho > 1$  be fixed. If  $\liminf_{s \rightarrow \infty} f_\lambda^*(1, s) > 0$  then  $\limsup_{s \rightarrow \infty} f_\lambda^*(\rho, s) > 0$ .*

A result along these lines is given by Bollobás and Riordan (Theorem 4.1 of [3]). The result in [3] is for Voronoi percolation but the proof can be transferred to our model, as we now discuss.

Much of the proof in [3] relies only on the Harris-FKG inequality, which holds in the present model as well (see Penrose and Sudbury [21]). These arguments in [3] carry over easily, making sure that rectangles with even integer sides are chosen as the RSA model is on a discrete lattice not a continuum.

In the first part of the proof in [3], an event  $E_{dense}$  is considered, and we need a different version of this event here. Given an integer  $s$  and constant  $\rho > 1$  let  $R_s$  be an  $s$  by  $\lfloor \rho s \rfloor$  rectangle. Given a rectangle  $R$  with integer sides  $a$  and  $b$  let  $R[r]$  be the rectangle with sides  $a + 2r$  and  $b + 2r$  centred on  $R$ , so the edges of  $R[r]$  are at distance  $r$  from the edges of  $R$ . Let  $E_{dense}(R_s)$  be the event that no site in  $R_s$  is affected by any site outside  $R_s[r - 1]$ , where we take  $r$  to be  $2\lfloor \sqrt{s} \rfloor$ . By a similar argument to the proof of Lemma 9.1, we have the following result, which is analogous to Lemma 2.3 of [3].

**Lemma 9.2** *Let  $\rho \geq 1$  be constant. Let  $R_s$  and  $E_{dense}(R_s)$  be described as above. Let  $r = 2\lfloor \sqrt{s} \rfloor$ . Then  $P[E_{dense}(R_s)] \rightarrow 1$  as  $s \rightarrow \infty$ . Also  $E_{dense}(R_s)$  depends only on the arrival times at sites in  $R_s[r]$ .*

**Proof.** Any path from outside  $R_s[r - 1]$  to the edge of  $R_s$  must end on the edge of  $R_s$  and have length at least  $r$ . The number of possible self-avoiding paths of length  $r$  ending at the edge of  $R_s$  is at most  $2(s + \lfloor \rho s \rfloor)3^r := Ks3^r$ . For any self-avoiding path of length  $r$ , taking alternate sites along the path one has at least  $\lfloor r/2 \rfloor$  independent identically distributed arrival times, so the probability they occur in increasing order is at most  $1/\lfloor r/2 \rfloor!$ . Any path from outside  $R_s[r - 1]$  to the edge of  $R_s$  must end with one of these paths of length  $r$  so the probability that  $E_{dense}(R_s)$  does not occur is at most  $\frac{Ks3^r}{\lfloor r/2 \rfloor!}$  which tends to 0 as  $s$  tends to  $\infty$ . If the event  $E_{dense}(R_s)$  occurs then nothing outside  $R_s[r]$  can affect the states of the sites inside  $R_s$ . □

To prove Proposition 9.2, assume for a contradiction that it does not hold and fix a value of  $\lambda$  where it fails. Then  $\liminf_{s \rightarrow \infty} f_\lambda^*(1, s) > 0$  and for some  $\rho > 1$  we have  $\lim_{s \rightarrow \infty} f_\lambda^*(\rho, s) = 0$ . But then, as in (4.4) of [3], for any  $\varepsilon > 0$  we have  $f_\lambda^*(1 + \varepsilon, s) \rightarrow 0$  as  $s$  goes to  $\infty$ . Throughout the argument let  $T_s$  be the strip  $[1, s] \times \mathbb{Z}$ . The first claim in the proof in [3] can easily be adapted to the integer lattice as follows.

**Lemma 9.3** *Let  $\varepsilon > 0$  be fixed and let  $\delta := \delta(s) := \frac{\lfloor \varepsilon s \rfloor}{s}$ . Let  $L$  be the line segment  $\{1\} \times [-\delta s, \delta s]$ . Then the probability that there is a black path  $P$  in  $T_s$  starting from  $L$  and going outside  $S' = [1, s] \times [-(1/2 + 2\delta)s, (1/2 + 2\delta)s]$  tends to zero as  $s \rightarrow \infty$ .*

**Proof.** By symmetry in the line  $[1, s] \times \{0\}$  it suffices to show that the event  $E$  that there is a black path  $P_1$  lying entirely within  $S'$  and connecting some site of  $L$  to some site at the top of  $S'$  has probability tending to zero.

Let  $E_1$  be the event that there is such a path  $P_1$  lying entirely in the rectangle  $R = [1, s] \times [-s/2, s/2 + 2\delta s]$ . If  $E$  holds but  $E_1$  does not then there is a black crossing the long way of an  $s$  by  $s + 2\delta s + 1$  rectangle which has probability tending to zero. Therefore it suffices to show that  $P(E_1) \rightarrow 0$ .

Reflecting vertically in the line  $y = \delta s$ , let  $L' := \{1\} \times [\delta s, 3\delta s]$  be the image of  $L$ . Let  $E_2$  be the event that there is a black path  $P_2$  from  $L'$  to some point with height  $-s/2$ . Then by symmetry and by the Harris-FKG inequality the probability that  $E_1$  and  $E_2$  occur is at least  $P(E_1)^2$ . But then  $P_1$  and  $P_2$  must meet and therefore contain a black path crossing  $R$  from top to bottom which has probability tending to zero, so  $P(E_1) \rightarrow 0$ .  $\square$

**Proof of Proposition 9.2.** The rest of the claims in [3] can be treated similarly by replacing squares in the plane with squares in the  $\mathbb{Z}^2$  lattice and making use of the Harris-FKG inequality holding in the RSA model. Then this combined with Lemma 9.2 and the fact that  $r = 2\lfloor \sqrt{s} \rfloor$  is  $o(s)$  completes the proof of Proposition 9.2.  $\square$

For  $n \in \mathbb{N}$  we define the boxes

$$B(2n+1) := [-n, n] \times [-n, n]; \quad B(2n) := [-n, n-1] \times [-n, n-1]. \quad (9.2)$$

**Proof of Theorem 8.1.** By Proposition 9.1, there exists  $\mu < 1$  such that (9.1) holds. Defining  $\delta := (1/3) \limsup_{s \rightarrow \infty} f_\mu^*(4, s)$ , we have by (9.1) and Proposition 9.2 that  $\delta > 0$ .

Thus we can find infinitely many even  $n$  such that the probability of a black crossing (including diagonals) the long way of a  $4n$  by  $n$  rectangle is at least  $2\delta$ . With any such even  $n$  we can find an odd  $m$  bigger than  $n$  such that a crossing the long way of a  $4n$  by  $n$  rectangle means that there is a crossing the long way of a  $3m$  by  $m$  rectangle. Therefore we can find infinitely many odd  $n$  such that there is a crossing of a  $3n$  by  $n$  rectangle with probability at least  $2\delta$ . Then for odd  $n$ , using the Harris-FKG inequality for this model (see [21]), the probability of there being a circuit of the annulus  $B(3n) \setminus B(n)$  is at least  $(2\delta)^4$ .

By Lemma 9.1, for any  $n$  we can find an  $m$  depending on  $n$  such that

$$P \left[ \bigcup_{y \in \mathbb{Z}^2 \cap B(n), z \in \mathbb{Z}^2 \setminus B(m)} (\{y \text{ affects } z\} \cup \{z \text{ affects } y\}) \right] \leq \delta^4.$$

Thus, we can build up a sequence  $m_1 < n_1 < m_2 < n_2 < \dots$  such that (i) for any  $i \in \mathbb{N}$ , the annulus  $A_i := B(3n_i) \setminus B(n_i)$  fits inside the annulus  $A'_i := B(m_{i+1}) \setminus B(m_i)$  and (ii) The probability that there exists any vertex inside  $A_i$  that is affected from outside  $A'_i$  is at most  $\delta^4$ .

Then let  $E_i$  be the event that (i) there is a closed circuit around the origin consisting of sites in the annulus  $A_i$  that are black for the process restricted to  $A'_i$  and (ii) no site of  $A_i$  is affected by any site outside  $A'_i$ . Then for all  $i$ ,  $P[E_i] \geq \delta^4$  and all the events  $E_i$  are independent. If any one of these events occurs there cannot be an infinite white component in  $\Lambda$  containing the origin, so by the Borel-Cantelli lemma the probability of an infinite white component occurring is 0. Therefore we have

$$\lambda_c \geq 1/\mu > 1$$

which completes the proof, subject to proving Proposition 9.1.  $\square$

### 9.3 Enhancement

We now define an enhancement that we shall use to interpolate between the RSA models on  $\Lambda$  and on  $\Lambda^*$ . Consider the infinite  $(4, 8^2)$  lattice (see Figure 9.1: we use terminology from [4], page 155), with faces divided into *octagons* and *diamonds*. The octagons are centred at the sites of  $\mathbb{Z}^2$ , and the diamonds are centred at the sites  $\{z' : z \in \mathbb{Z}^2\}$ , where we set  $z' := z + (1/2, 1/2)$  (we shall refer to sites  $z', z \in \mathbb{Z}^2$  as *diamond sites*).

Now consider a certain dependent face percolation model on the infinite  $(4, 8^2)$  lattice, in which each octagon is given the same colour (black or white) as the corresponding site in the random sequential adsorption model, and each of the diamonds is black with probability  $p$  (the *enhancement probability*) and white otherwise (independently of everything else). Thus  $p = 0$  is equivalent to  $\Lambda$  and  $p = 1$  is equivalent to  $\Lambda^*$ .

Placing a vertex at the centre of each face of the  $(4, 8^2)$  lattice, and taking two vertices to be adjacent if and only if the corresponding faces of the  $(4, 8^2)$  lattice are adjacent, we obtain the so-called *centred quadratic lattice* (see [13]), and we may equivalently view the dependent face percolation model just described as a site percolation model on the centred quadratic lattice.

Let  $h(n, \lambda, p)$  denote the probability that there is a horizontal black crossing in  $\Lambda$  of a  $2n$  by  $2n$  square  $B(2n)$  (as defined at (9.2)) with arrivals rate  $\lambda$  on the even sites and 1 on the odd sites and enhancement probability  $p$ . In this model we must have either a horizontal crossing or a vertical white crossing but not both. Also, for  $(\lambda, p) = (1, 0.5)$  the probability of both these events must be the same by symmetry so the probability of a horizontal black crossing is 0.5. That is, for any  $n$  we have

$$h(n, 1, 0.5) = 0.5. \quad (9.3)$$

**Remark 9.1** *By (9.3) and monotonicity, we have  $h(n, 1, 1) \geq 0.5$  and therefore (9.1) holds for  $\mu = 1$ . Hence, by the argument already given in the proof of Theorem 8.1 at the end of Section 9.2, we have  $\lambda_c \geq 1$ . The remainder of this section is concerned with demonstrating that this inequality is strict.*

To each diamond  $x', x \in \mathbb{Z}^2$ , we assign a uniform random variable  $T_{x'}$  (the *enhancement variable*). Then  $x'$  is black if  $T_{x'} < p$  and white otherwise. We then introduce the idea of a site being *pivotal*. Let  $H_n$  be the event that we have a horizontal crossing of  $B(2n)$  in the enhanced model on  $\Lambda$ . Then we say that an even site  $x$  is 1-pivotal if making the arrival time  $t_x$  equal to the first arrival of the Poisson process at  $x$  means that  $H_n$  occurs but making  $t_x$  equal to the second arrival time of the Poisson process at  $x$  means it does not. We say that a diamond  $x'$  is 2-pivotal if making  $T_{x'} = 0$  means  $H_n$  occurs but if  $T_{x'} = 1$  then it does not.

For  $x \in \mathbb{Z}^2$ , let  $P_1(n, \lambda, x)$  be the probability that site  $x$  is 1-pivotal, and let  $P_2(n, \lambda, x)$  be the probability that site  $x'$  is 2-pivotal. We have the following proposition (a variant of the Margulis-Russo formula).



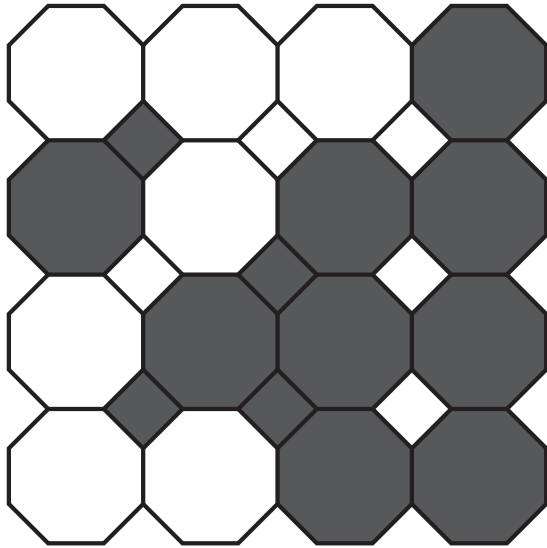
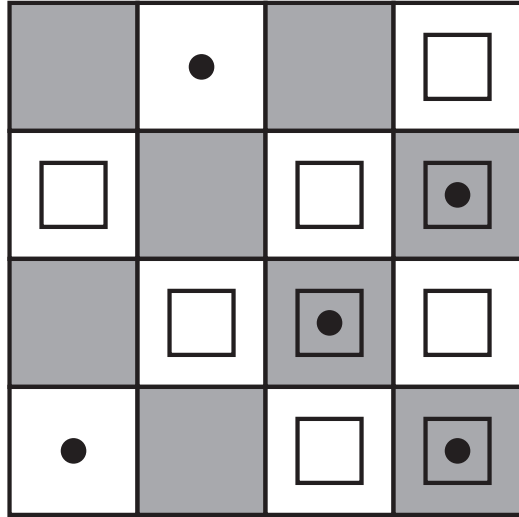


Figure 9.1: Here is an example of random sequential adsorption and a corresponding percolation process on the faces of the  $(4, 8^2)$  lattice.

**Proposition 9.3** *It is the case that*

$$\frac{\partial h(n, \lambda, p)}{\partial \lambda} = (1/\lambda) \sum_{x \in \mathbb{Z}^2: x \text{ even}} P_1(n, \lambda, x) \quad (9.4)$$

and

$$\frac{\partial h(n, \lambda, p)}{\partial p} = \sum_{x \in \mathbb{Z}^2} P_2(n, \lambda, x) \quad (9.5)$$

**Proof.** Fix  $n$  and  $p$ . Enumerate the even sites of  $\mathbb{Z}^2$  in some manner as  $x_1, x_2, \dots$ . Given  $k \in \mathbb{N}$  and given  $\lambda_1 > 0, \lambda_2 > 0$ , let  $E_k(\lambda_1, \lambda_2)$  be the event that  $H_n$  occurs when we use a Poisson arrivals process of rate  $\lambda_1$  at sites  $x_1, \dots, x_{k-1}$  and of rate  $\lambda_2$  at sites  $x_k, x_{k+1}, x_{k+2}, \dots$ . Let  $\varepsilon > 0$ . For  $x \in \mathbb{Z}^2$  let  $A(x)$  be the event that site  $x$  affects some site in  $B_n$ . Since the probability there is a path with increasing arrival times starting at  $x$  and ending at  $B_n$  decays at least exponentially in the distance from  $x$  to  $B_n$  (see the proof of Lemma 9.1), the sum  $\sum_{k \geq 1} P[A(x_k)]$  converges. Hence by the first Borel-Cantelli lemma,

$$0 \leq P[E_k(\lambda, \lambda + \varepsilon)] - h(n, \lambda, p) \leq P\left[\bigcup_{j=k}^{\infty} A(x_j)\right] \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence,

$$\begin{aligned} h(n, \lambda + \varepsilon, p) - h(n, \lambda, p) &= P[E_1(\lambda, \lambda + \varepsilon)] - \lim_{k \rightarrow \infty} P[E_k(\lambda, \lambda + \varepsilon)] \\ &= \sum_{k=1}^{\infty} P[E_k(\lambda, \lambda + \varepsilon) \setminus E_{k+1}(\lambda, \lambda + \varepsilon)]. \end{aligned} \quad (9.6)$$

Here we are assuming the Poisson processes of rate  $\lambda$  and  $\lambda + \varepsilon$  at  $x_k$  are coupled in the usual way, i.e. with the  $(\lambda + \varepsilon)$ -process decomposed into two independent processes of rate  $\lambda$  and  $\varepsilon$  respectively.

Event  $E_k(\lambda, \lambda + \varepsilon) \setminus E_{k+1}(\lambda, \lambda + \varepsilon)$  occurs if and only if (i) the first arrival time  $T_1$  of the  $(\lambda + \varepsilon)$ -process at  $x_k$  comes from the  $\varepsilon$ -process, and (ii) the crossing of  $B(2n)$  occurs if we use the arrival time  $T_1$  at  $x_k$ , but not if we use the arrival time  $T_1 + T_2$ , where  $T_2$  is the time from  $T_1$  to the next arrival of the  $\lambda$ -process at  $x_k$ . Note that  $T_2$  is exponential with parameter  $\lambda$ , independent of  $T_1$  and the type of the arrival at time  $T_1$ . Therefore,

$$P[E_k(\lambda, \lambda + \varepsilon) \setminus E_{k+1}(\lambda, \lambda + \varepsilon)] = (\varepsilon/(\lambda + \varepsilon))P[F_k(\lambda, \lambda + \varepsilon)] \quad (9.7)$$

where  $F_k$  denotes the event that the crossing of  $B_n$  occurs if we use the first arrival at  $x_k$  but not if we use the second arrival at  $x_k$ , and our arrivals processes are Poisson rate  $\lambda$  at sites  $x_j, j < k$ , and Poisson rate  $\lambda + \varepsilon$  at sites  $x_i, i > k$ , and our first arrival at  $x_k$  is exponential rate  $\lambda + \varepsilon$  but the time from the first arrival to the second arrival at  $x_k$  is exponential rate  $\lambda$ .

Coupling events  $F_k(\lambda, \lambda + \varepsilon)$  and  $F_k(\lambda, \lambda)$ , we have for any integer  $K > n$  that  $P[F_k(\lambda, \lambda + \varepsilon) \setminus F_k(\lambda, \lambda)]$  is bounded by the sum of the probability that there is some site inside  $B(2n)$  that is affected from outside  $B(2K)$ , and the probability that there exists some site  $x_j$  inside  $B(2K)$  such that the first arrival for the  $(\lambda + \varepsilon)$ -process at  $x_j$  comes from the  $\varepsilon$ -process at that site. For any fixed  $K$  the second of these probabilities tends to zero as  $\varepsilon \downarrow 0$ , while the first probability is small for large  $K$ , uniformly in  $\varepsilon$ . Hence by (9.7),

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} P[E_k(\lambda, \lambda + \varepsilon) \setminus E_{k+1}(\lambda, \lambda + \varepsilon)] = \lambda^{-1} P[F_k(\lambda, \lambda)] = \lambda^{-1} P_1(n, \lambda, x_k).$$

Moreover,  $P[F_k(\lambda, \lambda + \varepsilon)]$  is bounded by the probability that there are increasing arrival times along some path from  $x_k$  to  $B_n$ , which is bounded by a summable function of  $k$  uniformly in  $\varepsilon$ . Therefore by (9.6), (9.7) and dominated convergence we have

$$\frac{\partial^+ h}{\partial \lambda} = \lim_{\varepsilon \downarrow 0} \frac{h(n, \lambda + \varepsilon, p) - h(n, \lambda, p)}{\varepsilon} = \lambda^{-1} \sum_{k=1}^{\infty} P_1(n, \lambda, x_k). \quad (9.8)$$

By a similar argument (we omit details), one can obtain the same expression for the left derivative  $\frac{\partial^- h}{\partial \lambda}$ . Therefore (9.4) is proven.

The proof for the second part (9.5) is similar.  $\square$

## 9.4 Comparison of pivotal probabilities

The following proposition is a key step in the proof of Theorem 8.1.

**Proposition 9.4** *There exists a constant  $K_1 \in (0, \infty)$  such that for all  $n$ , all  $(\lambda, p) \in [0.5, 1.5] \times [0.2, 0.8]$ , and any even site  $y$  in  $B(2n)$ , there exists an adjacent diamond site  $\tilde{y}'$  such that*

$$P_1(n, \lambda, y) \leq K_1 P_2(n, \lambda, \tilde{y}').$$

The rest of this section is devoted to proving Proposition 9.4. The argument is quite lengthy and we divide it into stages.

Fix  $y \in \mathbb{Z}^2$ . For  $r, s \in \mathbb{N}$  with  $s > r$ , let  $C_r$  be the square of side  $2r + 1$  centred at  $y$ , and let  $A_{r,s} := C_s \setminus C_r$ .

We shall consider a coupling of RSA processes. Let  $S_x$  be the arrival times and enhancement variables in one process (so if  $x \in \mathbb{Z}^2$  then  $S_x$  is exponentially distributed but  $S_{x'}$  is a uniformly distributed enhancement variable). Let  $T_x$  be the arrival times and enhancement variables in another independent process. Given  $r, s \in \mathbb{N}$  with  $s \geq r$ , we use these to create a third process of arrival times and enhancement variables  $U_x^{(r,s)}$ , as follows. Put

$$U_x^{(r,s)} := \begin{cases} S_x, & x \notin C_s \\ \mathcal{B}_x S_x + (1 - \mathcal{B}_x) T_x, & x \in A_{r,s} \\ T_x, & x \in C_r \end{cases} \quad (9.9)$$

where the  $\mathcal{B}_x$  are independent Bernoulli variables with parameter 0.5.

The next lemma establishes a sort of conditional independence between the occupancy status, in the  $U_x^{(r,s)}$  process, of sites inside  $C_r$  and of sites outside  $C_s$ , conditional on the occurrence of a certain event associated with sites in the annulus  $A_{r-2,s}$ .

For  $x \in \mathbb{Z}^2$ , define  $I_S(x)$  to be 1 if site  $x$  is occupied and 0 if it is blocked in the  $S_x$  process. Define the following sets of sites:

$$M^{(r,s)} := \{x \in A_{r,s} \cap \mathbb{Z}^2 : I_S(x) = 1\}; \quad N^{(r,s)} := A_{r,s} \setminus M^{(r,s)}; \quad (9.10)$$

$$M_1^{(r,s)} := \{x \in M^{(r,s)} : S_x \leq 1\}; \quad M_2^{(r,s)} := M^{(r,s)} \setminus M_1^{(r,s)};$$

$$N_1^{(r,s)} := \{x \in N^{(r,s)} : S_x \leq 1\}; \quad N_2^{(r,s)} := N^{(r,s)} \setminus N_1^{(r,s)}.$$

Define the event

$$\begin{aligned} E_1^{(r,s)} := & \cap_{x \in M_1^{(r,s)} \cup N_2^{(r,s)}} \{\mathcal{B}_x = 1\} \cap \cap_{x \in M_2^{(r,s)} \cup N_1^{(r,s)}} \{\mathcal{B}_x = 0\} \\ & \cap \cap_{x \in M_2^{(r,s)}} \{T_x \leq 1\} \cap \cap_{x \in N_1^{(r,s)}} \{T_x > 1\} \\ & \cap \cap_{x \in M^{(r-2,r)}} \{T_x \leq 1\} \cap \cap_{x \in N^{(r-2,r)}} \{T_x > 1\}. \end{aligned} \quad (9.11)$$

**Lemma 9.4** *Suppose  $r, s \in \mathbb{N}$  with  $r \geq 3$  and  $s \geq r + 3$ . If  $E_1^{(r,s)}$  occurs then the state of all sites in  $\mathbb{Z}^2 \setminus C_r$  is the same in the  $U_x$  process as in the  $S_x$  process.*

**Proof.** Assume event  $E_1^{(r,s)}$  occurs. We start off with all the arrival times in the  $S_x$  process. Then we change the arrival times in  $M_2^{(r,s)}$  one by one. Each time we are making the arrival time at an occupied site earlier, so we cannot change the state of any sites. Then we change the arrival times in  $N_1^{(r,s)}$  one by one. Each time we are making the arrival time at a blocked site later so we cannot change the state of any site. We then have our  $U_x$  process on  $\mathbb{Z}^2 \setminus C_r$ .

Now we change the arrival times for the sites inside  $C_r$ . Every site  $x \in M^{(r-1,s-1)}$  has  $U_x^{(r,s)} \leq 1$  and has all its neighbours  $z$  with  $U_z^{(r,s)} > 1$ , so is occupied in the  $U^{(r,s)}$ -process. Also, every site  $z \in N^{(r,s-2)}$  has  $U_z^{(r,s)} > 1$  and has at least one occupied neighbour  $x$  with  $U_x^{(r,s)} \leq 1$ , so is vacant.

Thus when we change the arrival times for the sites inside  $C_r$ , the states of sites in  $A_{r,s-2}$  do not change and therefore the states of sites in  $\mathbb{Z}^2 \setminus C_{s-2}$  also do not change.

Hence, whatever arrival times we have on  $C_{r-2}$ , the states of the sites of  $\mathbb{Z}^2 \setminus C_r$  do not change, so they are the same in the  $U_x^{(r,s)}$  process as in the  $S_x$  process.  $\square$

We aim to prove Proposition 9.4, so let us assume  $y \in B(2n)$  and  $y$  is an even site. Let  $\tilde{y}'$  be the first diamond site adjacent to  $y$  that is contained in  $B(2n)$  working clockwise from the top right (so  $\tilde{y} = y$  if  $y$  is in the interior of  $B(2n)$ ). Let  $D_r$  be the diamond of sites that are at  $\ell_1$  distance  $r$  or less from  $y$ .

We shall say that  $y$  is  $(1, r)$ -pivotal for event  $H_n$  if changing  $t_y$  from the second Poisson arrival time to the first arrival time, and changing any affected sites within  $r$  steps of  $y$ , means that  $H_n$  occurs but changing only sites within  $r-1$  steps of  $y$  means  $H_n$  does not occur (by this we mean changing the 4 sites adjacent to  $y$  as appropriate as the first step then changing any sites adjacent to these as appropriate as the second step and so on). Define  $P_{1,r,n}(y)$  to be the probability that  $y$  is  $(1, r)$ -pivotal for  $H_n$ .

Given  $n$  and  $y$ , define event  $E(r)$ , for  $r \in \mathbb{N}$ , as follows. First suppose that  $r \leq n/5$  and the left and right endpoints of  $D_{r+7}$  lie in  $B(2n)$ . Then let  $E(r)$  be the event that we have black paths in  $B(2n)$  from each side of  $B(2n)$  up to  $\mathbb{Z}^2 \cap C_{r+7}$  but no black path from one side of  $B(2n)$  to the other avoiding  $C_{r+7}$ . Here we are using the second arrival time at  $y$ .

If  $r \leq n/5$ , and the left (respectively, right) endpoint of  $D_{r+7}$  lies outside  $B(2n)$ , then let  $E(r)$  be the event that we have a black path in  $B(2n)$  from

the right (respectively, left) side of  $B(2n)$  up to  $\mathbb{Z}^2 \cap C_{r+7}$ , but no black path in  $B(2n)$  from one side of  $B(2n)$  to the other avoiding  $C_{r+7}$ .

If  $r > n/5$  then we define  $E(r)$  to be the whole sample space, so that  $P[E(r)] = 1$ .

**Lemma 9.5** *There exists a constant  $K_2 \in (0, \infty)$  such that for all  $n, r \in \mathbb{N}$  and all even  $y \in \mathbb{Z}^2$ , we have*

$$P_{1,r,n}(y) \leq \frac{K_2^r P[E(r)]}{\lfloor r/2 \rfloor!}, \quad r \geq 20; \quad (9.12)$$

$$P_{1,r,n}(y) \leq P[E(r)], \quad r \leq 20. \quad (9.13)$$

Suppose  $y$  is  $(1, r)$ -pivotal. Then, after changing all sites affected up to  $r$  steps from  $y$  when we set  $t_y$  to be the first arrival time rather than the second arrival time, we obtain a black crossing of  $B(2n)$ . Any such crossing path must include at least one site in  $D_r$  (otherwise  $y$  would not be  $r$ -pivotal). Therefore event  $E(r)$  occurs. Since  $P[E(r)]$  is nondecreasing in  $r$ , this immediately gives us (9.13).

Now suppose  $r \geq 20$ . Let  $F(r)$  be the event that there is a self-avoiding path in  $\mathbb{Z}^2$  from  $y$  of length  $r$ , namely  $y_1, y_2, y_3, \dots, y_r$ , such that  $t_{y_1} < t_{y_2} < \dots < t_{y_r}$ . If  $y$  is  $(1, r)$ -pivotal then  $F(r)$  must occur, and hence

$$P_{1,r,n}(y) \leq P[E(r) \cap F(r)]. \quad (9.14)$$

Also, as in the proof of Lemma 9.1 we have

$$P[F(r)] \leq \frac{4(3^{r-1})}{\lfloor r/2 \rfloor!} \quad (9.15)$$

and  $F(r)$  depends only on the arrival times inside  $D_r$ . However, it is not independent of  $E(r)$ .

We now consider the independent families of arrival times  $(S_x)$  and  $(T_x)$ , and a coupled arrival time process  $U_x^{(r+2, r+5)}$  as defined by (9.9).

Let  $E^S$ , respectively  $E^U$ , be the event that  $E(r)$  occurs based on the  $S_x$  process, respectively the  $U_x^{(r+2, r+5)}$  process. Let  $F^S$ , respectively  $F^U$  be the event that  $F(r)$  occurs based on the  $S_x$  process, respectively the  $U_x^{(r+2, r+5)}$  process. Then, defining event  $A := E_1^{(r+2, r+5)}$  as given by (9.11), we have from Lemma 9.4 the event identity  $E^S \cap A = E^U \cap A$ . Hence,

$$\begin{aligned} P[E^S \cap F^S] P[A | E^S \cap F^S] &= P[E^S \cap F^S \cap A] \\ &= P[E^U \cap F^S \cap A] \\ &\leq P[E^U \cap F^S] = P[E^U] P[F^S]. \end{aligned}$$

Also, there is a constant  $K_3$  such that

$$P[A|E^S \cap F^S] \geq K_3^{-r}.$$

Combining these inequalities and using the fact that  $P[E^U] = P[E^S]$  yields

$$P[E^S \cap F^S] \leq K_3^r P[E^S] P[F^S]$$

and combined with (9.14) and (9.15) this gives us the desired result (9.12).  $\square$

Now, given  $r \geq 20$ , we consider for a while the process  $U_x := U_x^{(2r+6, 2r+10)}$  as defined by (9.9).

If  $C_{2r}$  is contained in  $B(2n)$  and  $r \leq n/5$  then let  $G_r$  be the octagonal region  $C_{2r} \cap D_{4r-10}$ , a sort of truncated square. Note that each of the inner diagonal boundaries of  $G_r$  consists of odd sites and is of length 10. The exact length is not important; we just need a reasonably large separation between each corner of the octagon  $G_r$ . Let  $G_r^-$  be the slightly smaller octagonal region  $C_{2r-4} \cap D_{4r-14}$ .

If  $C_{2r}$  intersects one or more sides of  $B(2n)$  and  $r \leq n/5$  then let  $G_r$  be as above but take out a triangle of sites of height 9 or 10 where the octagon meets the edge of  $B_n$  in such a way that the inner boundary of what is left on the diagonal consists of odd sites. Also take out all sites above or to the side of the triangle that are outside  $B(2n)$ . (See Figure 9.2). Let  $G_r^-$  be the all sites in  $G_r$  that are not within 4 of being outside  $G_r$ .

If  $r > n/5$  then let  $G_r$  be  $B(2n+4)$ .

**Lemma 9.6** *There exists a constant  $\beta \in (0, \infty)$  with the following property. Given  $r \geq 20$ , if the event  $E(r)$  occurs in the  $S_x$  process, then there exists a stable set  $Q_1 \subset G_r \cap \mathbb{Z}^2$  having no element adjacent to the occupied  $\mathbb{Z}^2$  sites of the  $S_x$  process outside  $G_r$ , and disjoint sets  $Q_2, Q_3$  of diamond sites inside  $G_r$ , such that (i) each of  $Q_1, Q_2, Q_3$  has at most  $\beta r$  elements, and (ii) if, in the  $U_x$  process, all the sites in  $Q_1$  are occupied, all diamonds in  $Q_2$  are black, all the diamonds in  $Q_3$  are white, and (if  $r \leq n/5$ ) all sites in  $C_{2r+6} \setminus G_r$  are in the same state as for the  $S_x$  process, then  $\tilde{y}'$  is 2-pivotal for the  $U_x$  process.*

**Proof.** First suppose  $r \leq n/5$ . Since  $E(r)$  occurs, there must be disjoint black paths in the  $S_x$  process up to  $\mathbb{Z}^2 \cap C_{r+7}$  from each side of  $B(2n)$ . The strategy of the proof is to extend these paths in towards  $y$  while keeping them disjoint in order to make  $\tilde{y}'$  2-pivotal.

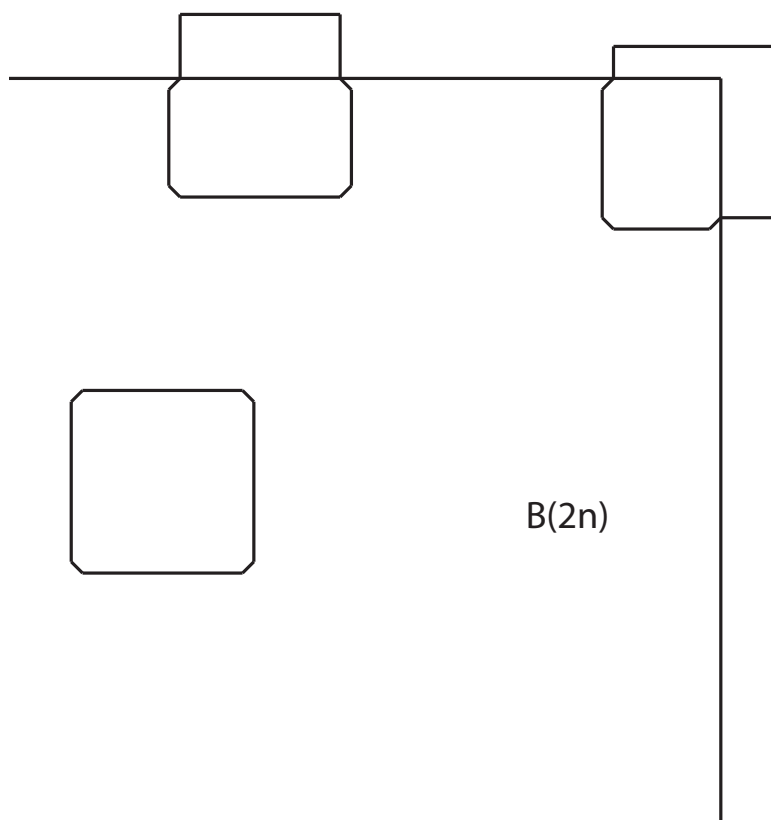


Figure 9.2: Examples of shapes  $G_r$  depending on where it intersects the edge of  $B(2n)$ .



For now we assume  $C_{2r}$  (and hence  $G_r$ ) is contained in  $B(2n)$  (so that  $\tilde{y} = y$ ). Let  $V$  be the set of black vertices (for the  $S_x$  process) in  $B(2n) \setminus G_r$  that are connected to the left hand side of  $B(2n)$  by a black path of the  $S_x$  process, without using any sites in  $G_r$ . Let  $v$  be the first even site inside  $G_r$  (according to the lexicographic ordering) that is occupied (for the  $S_x$  process) and connects to  $V$  either directly or via blocked odd sites adjacent to itself and  $V$  (and possibly also a black diamond site). Let  $W$  be the set of black sites (for the  $S_x$  process) in  $B(2n) \setminus G_r$  that are connected to the right hand side of  $B(2n)$  by a black path of the  $S_x$  process that avoids  $G_r$ . Let  $w$  be the first even site that is occupied inside  $G_r$  and connects to  $W$ . We now try and build paths from  $v$  and  $w$  in towards  $y'$  to make it 2-pivotal. We consider various cases of where  $v$  and  $w$  are:

**Case 1:** Suppose  $v$  and  $w$  are well away from each other. In this case we can always make  $y'$  2-pivotal. For example, if  $v$  and  $w$  are as in Figure 9.3, we can form disjoint paths  $P_1, P_2$  of even sites in towards  $y$ . In this and subsequent diagrams, the chequerboard squares are centred at sites of  $\mathbb{Z}^2$  and are shaded for even sites. Let  $I$  be the set of even sites  $\{v, w\} \cup P_1 \cup P_2$ . Let  $J$  be the set of odd sites in  $G_r \setminus G_r^-$  that are not adjacent to any site in  $I$  or to any of the occupied sites in  $C_{2r+6} \setminus G_r$ . Let  $J'$  be the set of odd sites in  $G_r^-$  that are three steps (in  $\mathbb{Z}^2$ ) away from  $I$ . Set  $Q_1 := I \cup J \cup J'$ . If the sites in  $Q_1$  are occupied for the  $T_x$  process, then  $y$  is 2-pivotal. The number of sites in  $Q_1$  is bounded by a constant times  $r$ .

In general, if we have  $v$  on a horizontal or vertical edge of  $G_r$ , then (see Figure 9.4) we can make the even site at position  $A$  in relation to  $v$  occupied to start  $P_1$ , switch the enhancement on at  $C'$  and due to the odd sites labelled  $B$  being occupied this cannot complete a crossing of  $B(2n)$ .

If  $v$  lies beside a diagonal edge of  $G_r$ , then (see Figure 9.5) we can make the even site at position  $A$  in relation to  $v$  occupied to start  $P_1$ , switch the enhancement on at  $C'$  and due to the odd sites labelled  $B$  being occupied this cannot complete a crossing of  $B(2n)$ .

**Case 2:** Suppose  $v$  and  $w$  are near each other but on a straight edge. If their columns are at distance 4 or more from each other and neither is in position  $I$  (see Figure 9.8) then there is no problem. Their columns cannot be at distance 2 from each other as then  $v$  and  $w$  would be connected to each other via black sites. If they are at distance 3 then there is no problem as long as neither  $v$  nor  $w$  is at position  $I$ . We have the enhancement switched

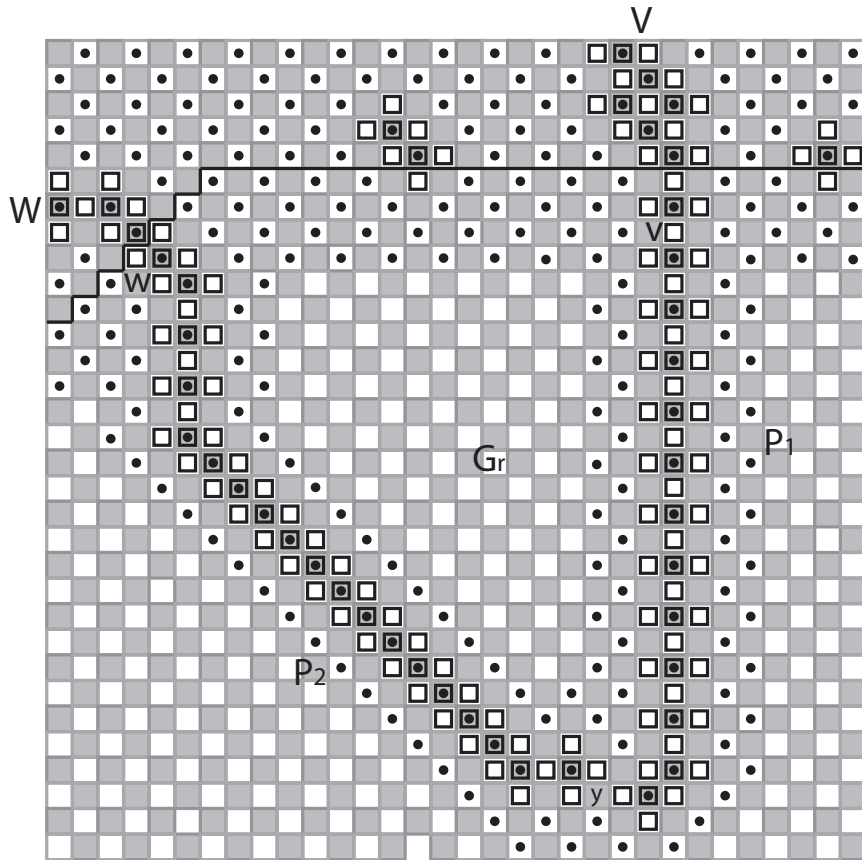


Figure 9.3: Construction of paths  $P_1, P_2$  making  $y'$  2-pivotal.

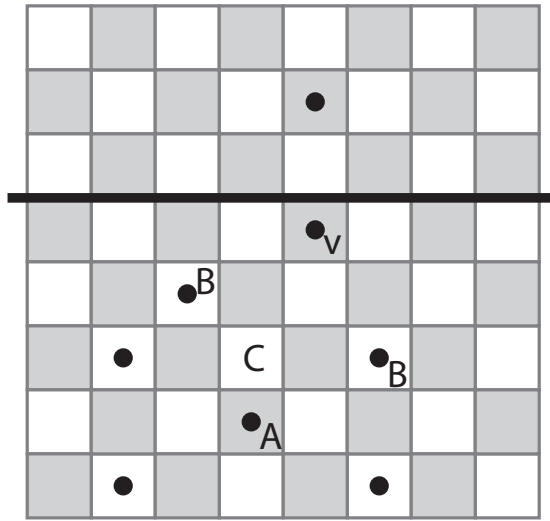


Figure 9.4: Starting path  $P_1$  when  $v$  is on a horizontal edge on the inner perimeter of  $G_r$ .

off at  $D$  (see Figure 9.6) and then extend the paths in towards  $y$ .

**Case 3:** Now suppose  $v$  and  $w$  are near each other on a diagonal edge. If their diagonals are at distance 3 there is no problem. They cannot be at distance 1 as then they would be connected. If they are at distance 2 and neither is at  $J$  there is no problem. We have the enhancement switched off at  $D$  (see Figure 9.7) and switched on at  $F$ .

**Case 4:** Suppose  $v$  and  $w$  lie near to each other but on a corner. We need to consider possible cases when  $v$  is at  $I$  or  $J$  (see Figure 9.8).

(a)  $v$  is at  $J$ . If  $w$  is 3 or more diagonals away then there is no problem. If  $w$  is 4 or more columns away then there is no problem. This just leaves three possibilities.

(i)  $w$  is at  $M$  (of Figure 9.8). Then refer to Figure 9.9. We can have an occupied even site at  $E$ , connected to  $v$  via a diamond site. There is no problem unless there is an occupied even site at  $A$  that is in  $W$ . Then we need to have an occupied odd site at  $D$  and have the enhancement at  $F'$  switched off. We can make  $D$  occupied because we know  $B$  is unoccupied since otherwise it would connect to both  $v$  and  $W$ .

(ii)  $w$  is at  $L$  of Figure 9.8. In this case, refer to Figure 9.10. We can have  $w$  connected to  $A$  and  $v$  connected to  $B$ , both via enhanced diamond sites, with the enhancement at  $C'$  switched off.

(iii)  $w$  is at  $K$  of Figure 9.8. Then refer to Figure 9.11. We aim to have an occupied site at  $E$  connected to  $v$ . This is fine as long as there is no site of  $W$  at  $B$  or  $C$ . If there is one at  $C$  but not  $B$  then we need to have an occupied odd site at  $A$  and switch off the enhancement which we can do as we know there is no occupied site at  $D$  as it would be joined to  $v$  and  $W$ . If there is a site of  $W$  at  $B$  then it is not actually possible to have  $y$  being  $r$ -pivotal as there is no way to get a path from  $V$  into  $D_r$  without joining up with  $W$  (which contradicts the event  $E(r)$  occurring in the  $S_x$  process). This is because  $v$  is blocked from having a path further into  $G_r$ , and there cannot be any other point in  $G_r$  connected to  $V$  elsewhere, because the paths in  $W$  from locations in  $G_r$  on both sides of  $v$  cut  $v$  off from being path-connected to any other part of the boundary of  $G_r$ .

(b)  $v$  is at  $I$  of Figure 9.8. If  $w$  is 3 or more diagonals away then there is no problem. If  $w$  is 4 or more columns away then there is no problem. This just leaves two possibilities.

(i) If  $w$  is at  $O$  of Figure 9.8, then (see Figure 9.12) this is akin to case (a) (iii) but just translated.

(ii) If  $w$  is at  $N$  of Figure 9.8, then (see Figure 9.13) we aim to have an occupied even site at  $A$ . We can do this unless there is an occupied site at  $B$  which is in  $W$ . If this happens then we aim for an occupied even site at  $E$  instead. This works so long as there is no occupied site at  $C$  in  $W$ . So there is no problem unless there are occupied sites at both  $B$  and  $C$  in  $W$ . If this happens then it is not actually possible to have  $y$  being  $r$ -pivotal as there is no way to get a path from  $V$  into  $D_r$  without joining up with  $W$ .

Now consider the cases where  $C_{2r}$  is not contained in  $B(2n)$ . First we look at the case where  $C_{2r}$  intersects just the top edge of  $B(2n)$ . Remember  $G_r$  is now as in Figure 9.2 and the triangular regions are of height 9 or 10, chosen in such a way that the inner boundary consists of odd sites. We then argue as before.. We have the sets  $V$  and  $W$  as before and the sites  $v$  and  $w$ . If  $v$  and  $w$  are both well away from the edge of  $B(2n)$  then we just have one of the cases we have already looked at. So we just consider the case where  $v$  say is near the edge of  $B(2n)$ . However as it is on a diagonal of  $G_r$  we can treat it as before and the path we create will stay inside  $B(2n)$ .

Now consider the case where  $C_{2r}$  intersects the right hand edge of  $B(2n)$ . In this case we just look at the set  $V$  and site  $v$  inside  $G_r$  that is connected to the left of  $B(2n)$ . Inside  $G_r$  we can then form a path from  $v$  towards  $y$  and a disjoint path from the right hand edge of  $B(2n)$  towards  $y$  and ensure that  $\tilde{y}'$  is 2-pivotal.

Finally we consider the case with  $r > n/5$ . In this case, we can make a path of even sites in from each boundary of  $B(2n)$  to  $y$ , together with a path of odd sites around the edge of each of these paths and around the boundary of  $B_n$ .  $\square$

**Proof of Proposition 9.4.** Assume  $(\lambda, p) \in [0.5, 1.5] \times [0.2, 0.8]$ . Suppose  $E(r)$  occurs for the  $S_x$  process. Let the sets  $Q_1, Q_2, Q_3$  be as in Lemma 9.6. Suppose also that  $E_1^{(2r+6, 2r+10)}$  occurs, and we have  $T_x \leq 1$  on all occupied sites (for the  $S_x$ -process) in  $C_{2r+4} \setminus G_r$  and  $T_x > 1$  on all blocked sites (for the  $S_x$ -process) in  $C_{2r+4} \setminus G_r$  (this is consistent with occurrence of event  $E_1^{(2r+6, 2r+10)}$ .) Suppose also that  $T_x \leq 1$  for all the sites in  $Q_1$  and  $T_x > 1$  on all the sites in  $\mathbb{Z}^2$  lying adjacent to  $Q_1$ , and  $T_{x'} < p$  for  $x' \in Q_2$  and  $T_{x'} > p$  for  $x' \in Q_3$ . Then using Lemma 9.4 we have that  $y$  is 2-pivotal for the  $U_x$  process. This all occurs with probability at least  $K_4^{-r}$  (given  $E(r)$ ), for some

finite positive constant  $K_4$ . Therefore for all  $y \in \mathbb{Z}^2 \cap B(n)$  and all  $r \geq 20$  we have that

$$P_2(n, \lambda, \tilde{y}) \geq K_4^{-r} P[E(r)]. \quad (9.16)$$

Hence by (9.12) and (9.13),

$$\begin{aligned} P_1(n, \lambda, y) &= \sum_{r=0}^{\infty} P_{1,r,n}(y) \leq 20P[E(20)] + \sum_{r=20}^{\infty} \frac{K_2^r P[E(r)]}{[r/2]!} \\ &\leq 20K_4^{20} P_2(n, \lambda, \tilde{y}) + \sum_{r=20}^{\infty} \frac{(K_2 K_4)^r P_2(n, \lambda, \tilde{y})}{[r/2]!} = K_1 P_2(n, \lambda, \tilde{y}), \end{aligned} \quad (9.17)$$

where  $K_1$  is a finite constant independent of  $\lambda$  and  $p$ , as required.  $\square$

## 9.5 Proof of Theorem 8.1

In the preceding section we found a lower bound for  $P_1(n, \lambda, y)$  in terms of  $P_2(n, \lambda, \tilde{y})$ , for  $y$  inside  $B(2n)$ . We now find a lower bound for  $P_1(n, \lambda, y)$  in terms of  $P_1(n, \lambda, z)$  for  $y$  outside  $B(2n)$  and  $z$  inside  $B(2n)$ . Once we have this, we shall be able to quickly complete the proof of Theorem 8.1.

We introduce more notation. Let  $\partial B(2n)$  be the set of even sites on the inner boundary of  $B(2n)$ . For  $x \in \mathbb{Z}^2 \setminus B(2n)$ , let  $z(x)$  be the nearest site in  $\partial B(2n)$  to  $x$  (here using graph distance in  $\mathbb{Z}^2$  as our measure of distance). If there is a choice of two we take  $z(x)$  to be the one clockwise from the other. For  $z \in B(2n)$ , set  $L_z := \{x \in \mathbb{Z}^2 \setminus B(2n) : z(x) = z\}$ .

**Proposition 9.5** *There exists a constant  $K_5$  such that for any  $(\lambda, p) \in [0.5, 1.5] \times [0.2, 0.8]$  and any  $z \in \partial B(2n)$  and even  $y \in L_z$  we have that*

$$P_{1,r,n}(y) \leq \frac{K_5^r P_1(n, \lambda, z) I_r(y)}{[r/2]!}, \quad r \geq 20; \quad (9.18)$$

$$P_{1,r,n}(y) \leq K_5 P_1(n, \lambda, z) I_r(y), \quad r \leq 20. \quad (9.19)$$

where  $I_r(y) = 1$  if  $y$  is within  $r$  steps of  $B(2n)$  and  $I_r(y) = 0$  otherwise.

**Proof.** Assume  $y$  is within  $r$  steps of  $B(2n)$ ; otherwise it cannot possibly be  $(1, r)$ -pivot. The proof is very similar to that of Proposition 9.4. We couple processes as before. That is, we start with independent  $S_x$  and  $T_x$  arrivals processes, and define  $U_x = U_x^{(2r+6, 2r+10)}$  by (9.9) as before.

Given  $y$ , and given  $r \in \mathbb{N}$ , define event  $E(r)$  as in Section 9.4. Although now  $y$  lies outside  $B(2n)$ , Lemma 9.5 remains valid.

Let event  $E_1 := E_1^{(2r+6, 2r+10)}$  be defined by (9.11) as before. By Lemma 9.4, the state of all sites in  $\mathbb{Z}^2 \setminus C_{2r+6}$  will be the same in the  $U_x$  process as in the  $S_x$  process if  $E_1$  occurs.

Define the region  $G_r$  as we did in the proof of Lemma 9.6 when there were boundary effects. Then  $z$  will lie in the region  $G_r$ .

Using Lemma 9.5 and Lemma 9.7 below, which is analogous to Lemma 9.6, as in the proof of Proposition 9.4, we can find a constant  $K_6$  such that for  $r \geq 20$  we have

$$P_{1,r,n}(y) \leq \frac{K_2^r P[E(r)]}{\lfloor r/2 \rfloor!} \leq \frac{K_2^r K_6^r P_1(n, \lambda, z)}{\lfloor r/2 \rfloor!}$$

which demonstrates (9.18). In the case with  $r \leq 20$ , we use Lemmas 9.5 and 9.7 to obtain

$$P_{1,r,n}(y) \leq P[E(r)] \leq P[E(20)] \leq K_6^{20} P_1(n, \lambda, z),$$

yielding (9.19). □

**Lemma 9.7** *There exists a constant  $\alpha \in (0, \infty)$  with the following property. Let  $y, z$  be as above and assume  $r \geq 20$ . If the event  $E(r)$  occurs in the  $S_x$  process, then there exists a stable set  $Q_1 \subset G_r \cap \mathbb{Z}^2$  having no element adjacent to the occupied sites of the  $S_x$  process outside  $G_r$ , and disjoint sets  $Q_2, Q_3$  of diamond sites inside  $G_r$ , such that (i) each of  $Q_1, Q_2, Q_3$  has at most  $\alpha r$  elements, and (ii) if, in the  $U_x$  process, all the sites in  $Q_1$  are occupied, all diamonds in  $Q_2$  are black, all the diamonds in  $Q_3$  are white, and all sites in  $C_{2r+6} \setminus G_r$  are in the same state as for the  $S_x$  process, then  $z$  is 1-pivotal for the  $U_x$  process.*

**Proof.** Suppose  $C_{2r}$  does not meet the left or right boundary of  $B(2n)$ . If  $E(r)$  occurs there must be disjoint paths in the  $S_x$  process up to  $\mathbb{Z}^2 \cap C_{r+7}$  within  $B(2n)$  from each side of  $B(2n)$ . By similar arguments to those in the proof of Lemma 9.6, we can obtain the event that  $z$  is 1-pivotal for the  $U_x$  process, by specifying  $O(r)$  vertices to be occupied.

Suppose  $C_{2r}$  meets the right boundary of  $B(2n)$ . Then if  $E(r)$  occurs there must be a path in the  $S_x$  process up to  $C_{r+7}$  within  $B(2n)$  from the

left side of  $B(2n)$ . Hence there is such path from the left boundary of  $B(2n)$  to the boundary of  $G_r$ . By similar arguments to before, we can obtain the event that  $z$  is 1-pivotal for the  $U_x$  process, by specifying  $O(r)$  vertices to be occupied so as to extend the existing path to  $z$ , and creating a disjoint path from the right hand edge of  $B(2n)$  to  $z$ .

The case where  $C_{2r}$  meets the left boundary of  $B(2n)$  is treated analogously.  $\square$

**Proof of Proposition 9.1.** By Proposition 9.5, there are constants  $K_7, K_8$  such that for any  $z \in \partial B(2n)$ ,

$$\begin{aligned} \sum_{y \in L_z: y \text{ even}} P_1(n, \lambda, y) &= \sum_{y \in L_z} \sum_{r=0}^{\infty} P_{1,r,n}(y) \\ &\leq \sum_{r=0}^{19} K_7 P_1(n, \lambda, z) + \sum_{r=20}^{\infty} \frac{P_1(n, \lambda, z) K_5^r K_7 r^2}{[r/2]!} \leq K_8 P_1(n, \lambda, z). \end{aligned}$$

Summing over  $z \in \partial B(2n)$ , we obtain that

$$\sum_{y \in \mathbb{Z}^2 \setminus B(2n): y \text{ even}} P_1(n, \lambda, y) \leq K_8 \sum_{z \in B(2n) \cap \mathbb{Z}^2: z \text{ even}} P_1(n, \lambda, z).$$

Putting this together with Proposition 9.4 gives for some  $K_9$  that

$$\sum_{y \in \mathbb{Z}^2: y \text{ even}} P_1(n, \lambda, y) \leq K_9 \sum_{z \in \mathbb{Z}^2: z' \in B(2n)} P_2(n, \lambda, z).$$

Hence by Proposition 9.3,

$$\frac{\partial h(n, \lambda, p)}{\partial \lambda} \leq K_9 \frac{\partial h(n, \lambda, p)}{\partial p}, \quad (\lambda, p) \in [0.5, 1.5] \times [0.2, 0.8].$$

We also know from (9.3) that  $h(n, 1, 0.5) = 0.5$ , so looking at a small box around  $(1, 0.5)$  we can find  $\varepsilon > 0$  such that for all  $n$ , we have  $h(n, 1 - \varepsilon, 1) \geq h(n, 1, 0.5) = 0.5$ . Therefore taking  $\mu = 1 - \varepsilon$  we have satisfied (9.1).  $\square$

With Proposition 9.1 proven, our proof of Theorem 8.1 is now complete by the arguments in Section 9.2 and the proof of the lower bound in Section 9.1.



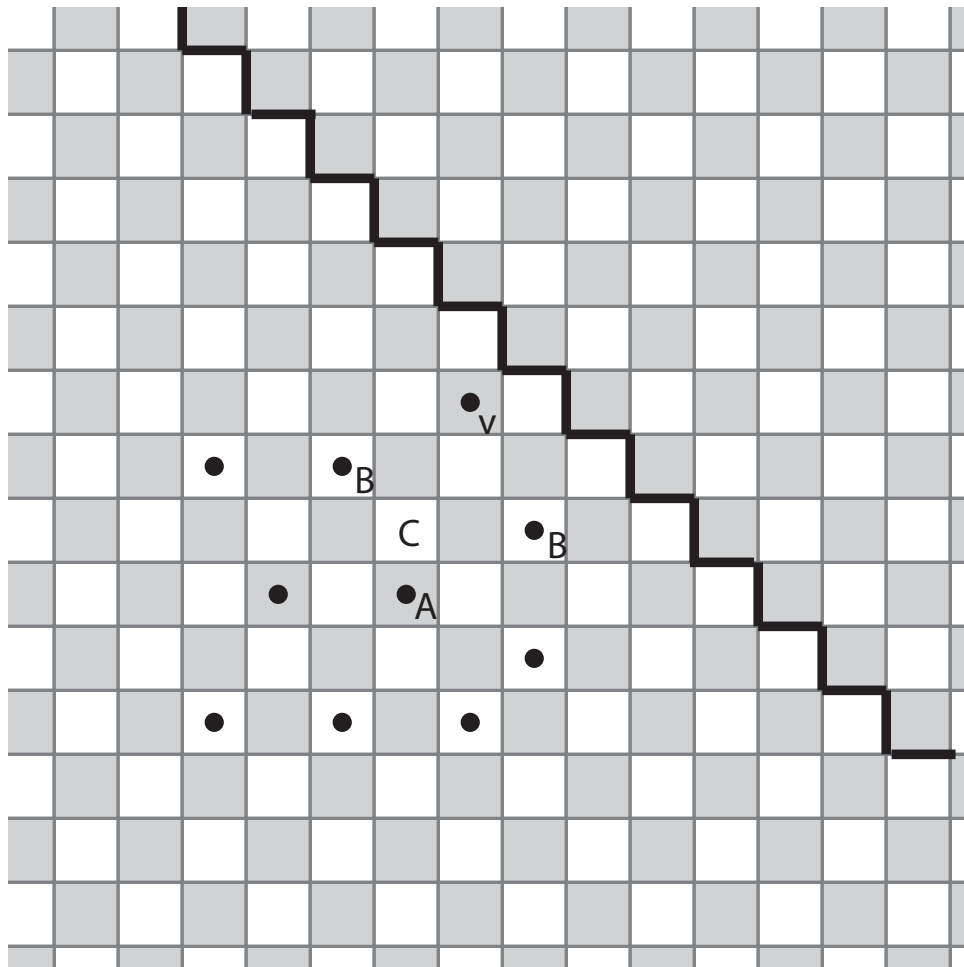


Figure 9.5: Starting the path  $P_1$  when  $v$  lies near a diagonal edge.

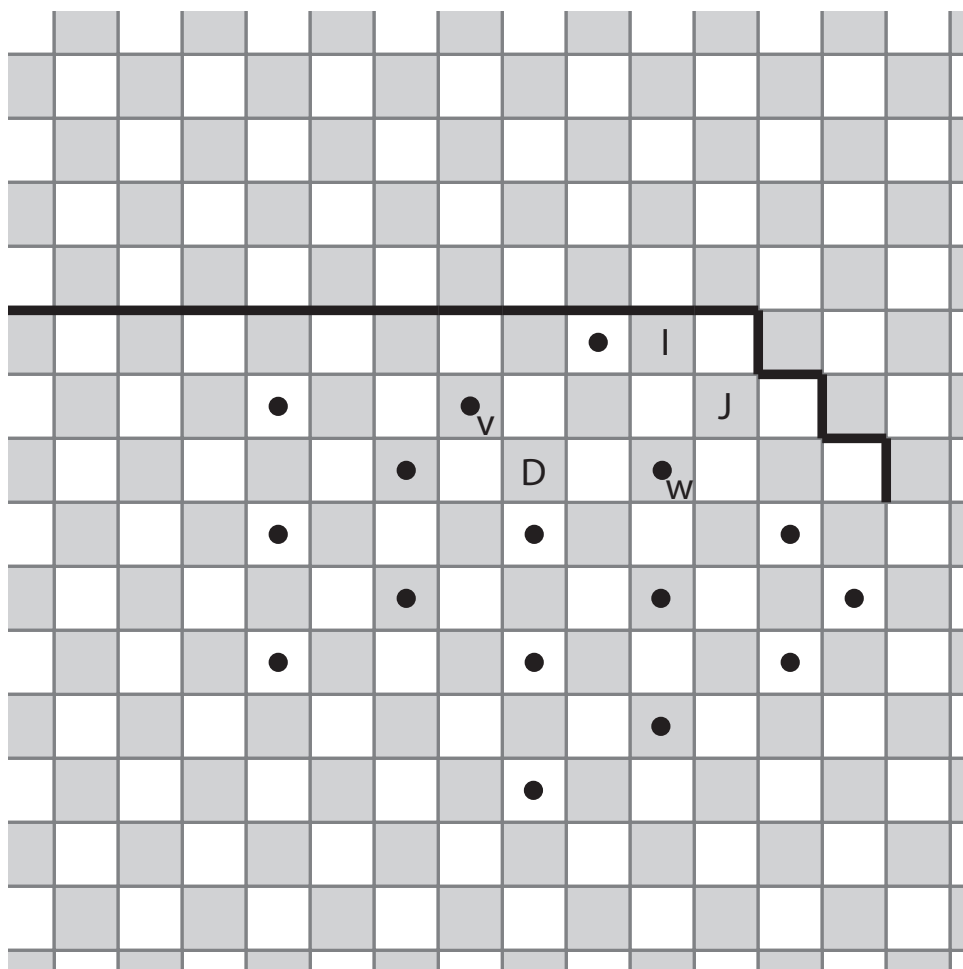


Figure 9.6: Case 2.

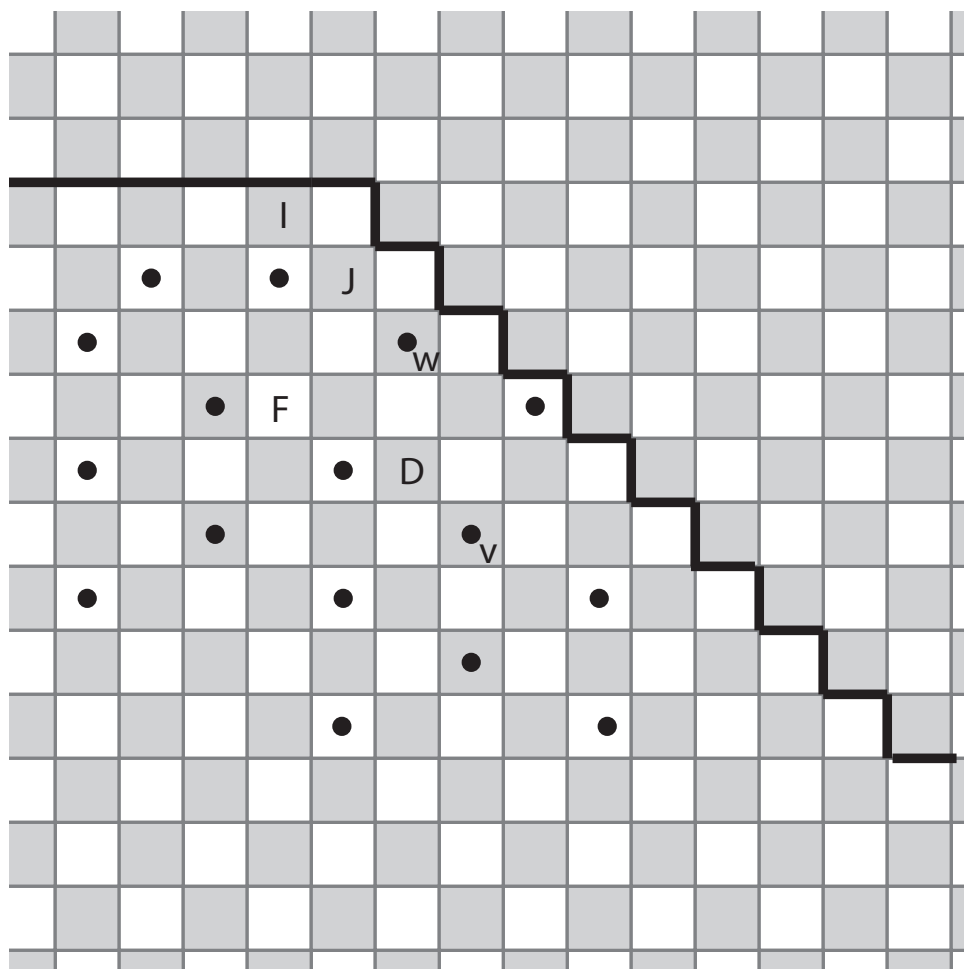


Figure 9.7: Case 3.

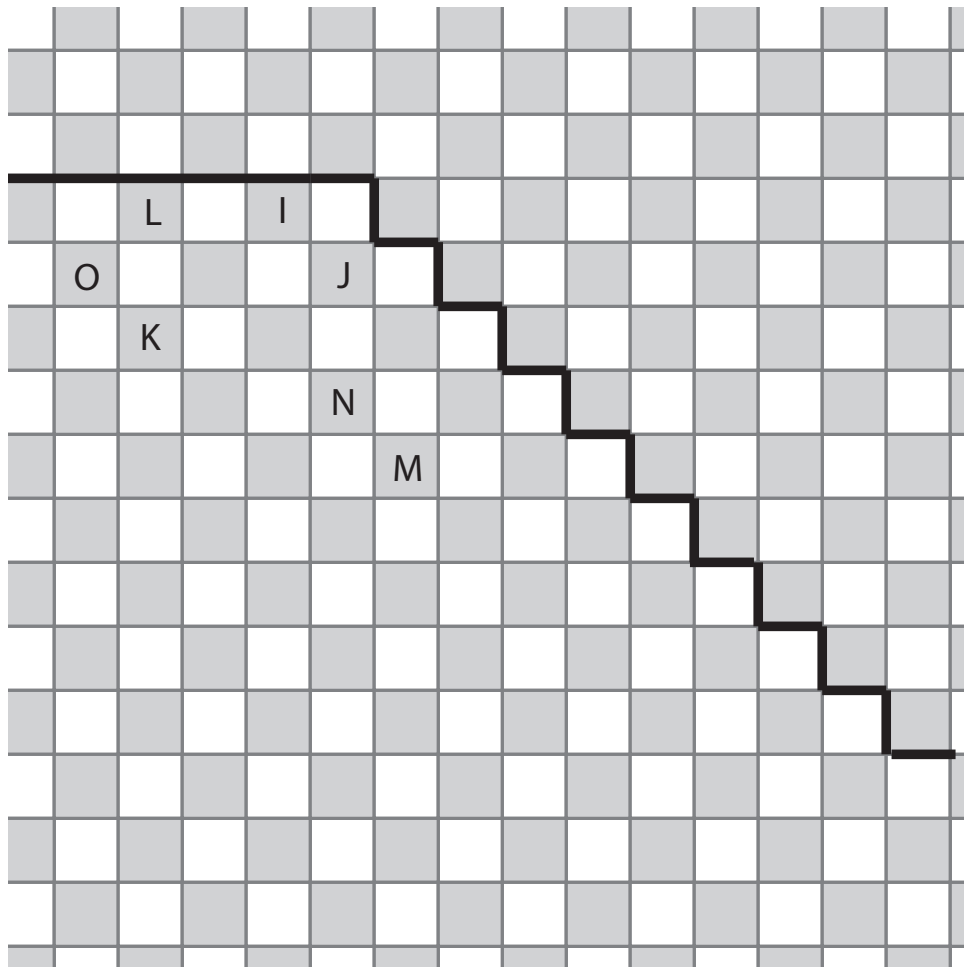


Figure 9.8: Identifying locations near a corner.

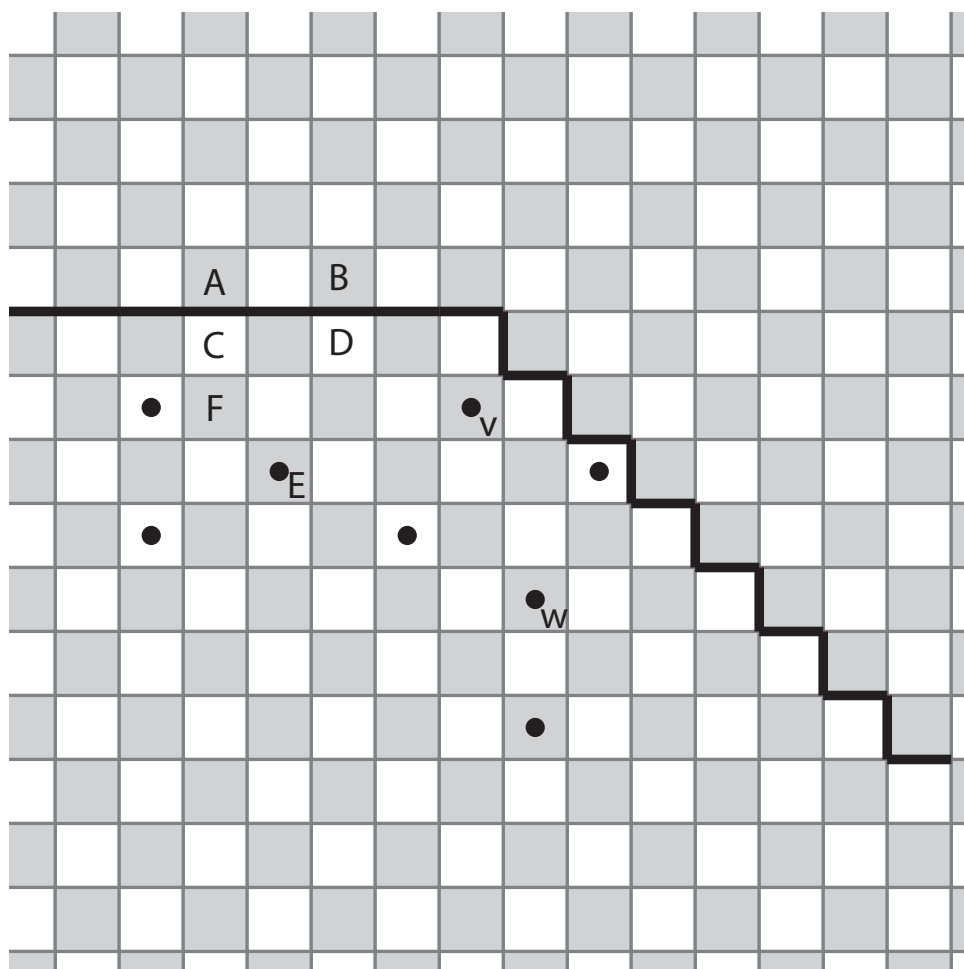


Figure 9.9: Case 4 (a) (i).

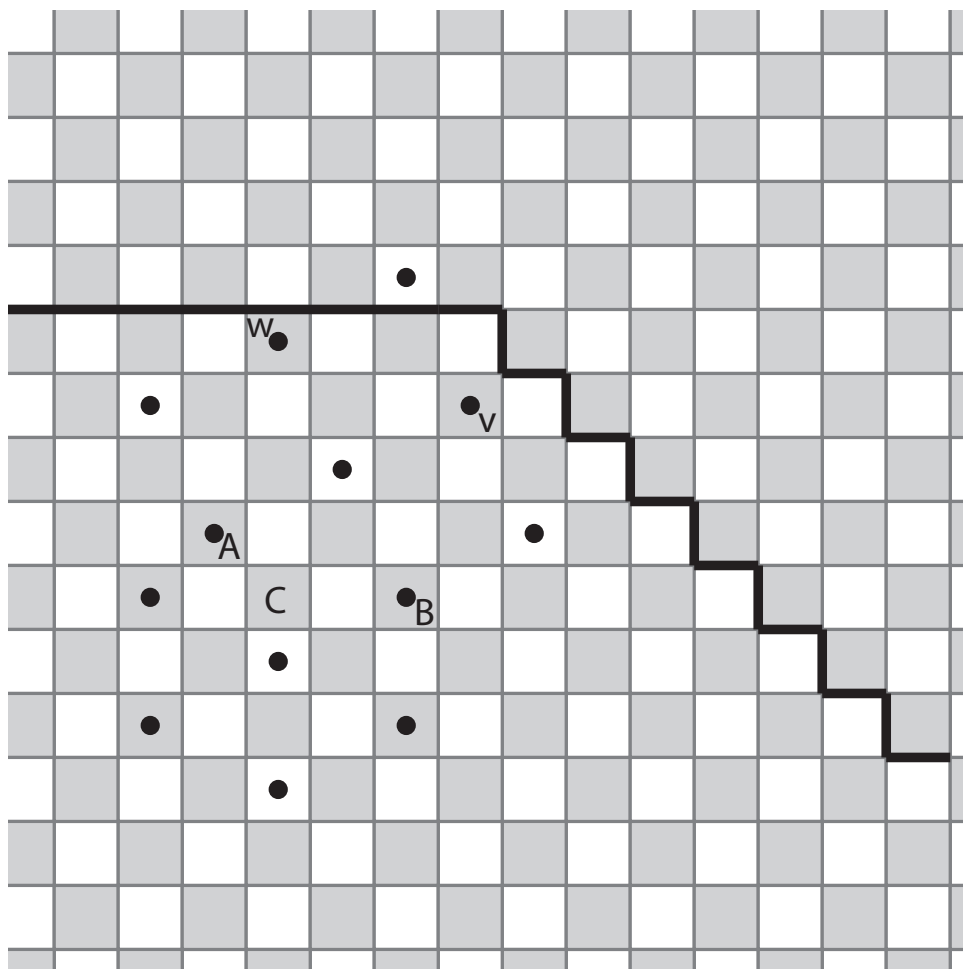


Figure 9.10: Case 4 (a) (ii).

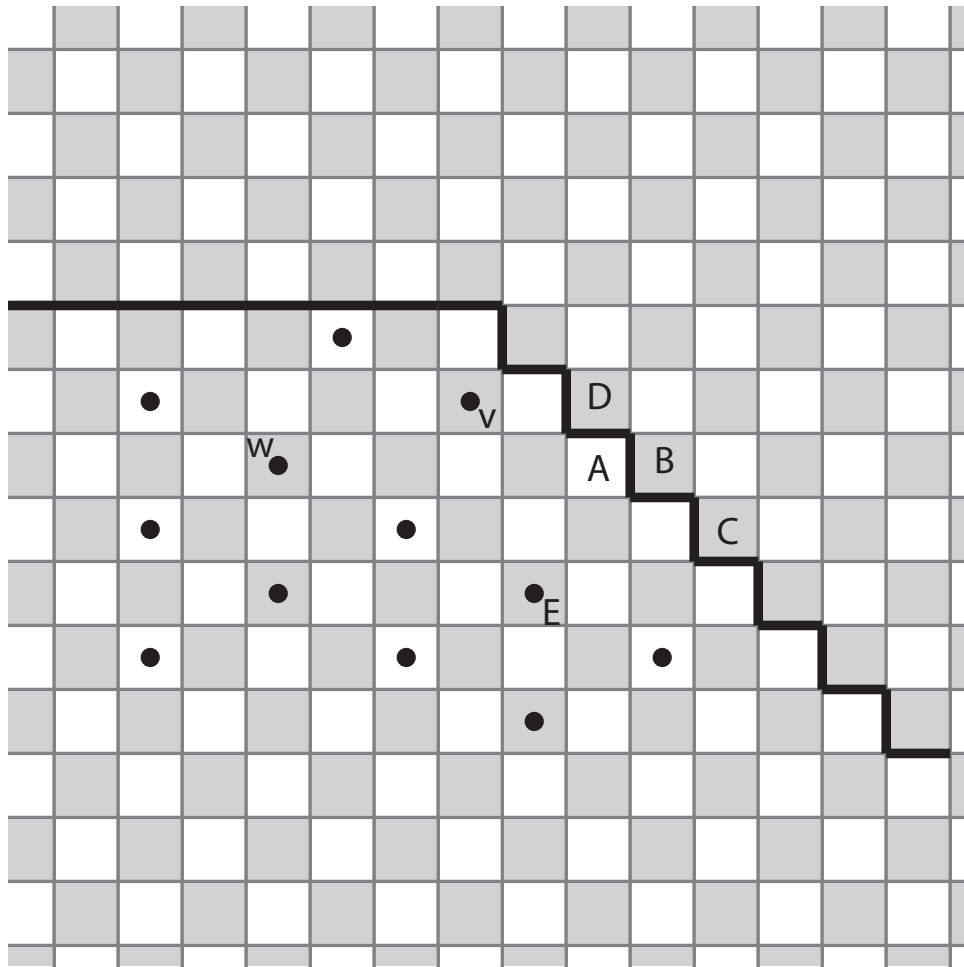


Figure 9.11: Case 4 (a) (iii).

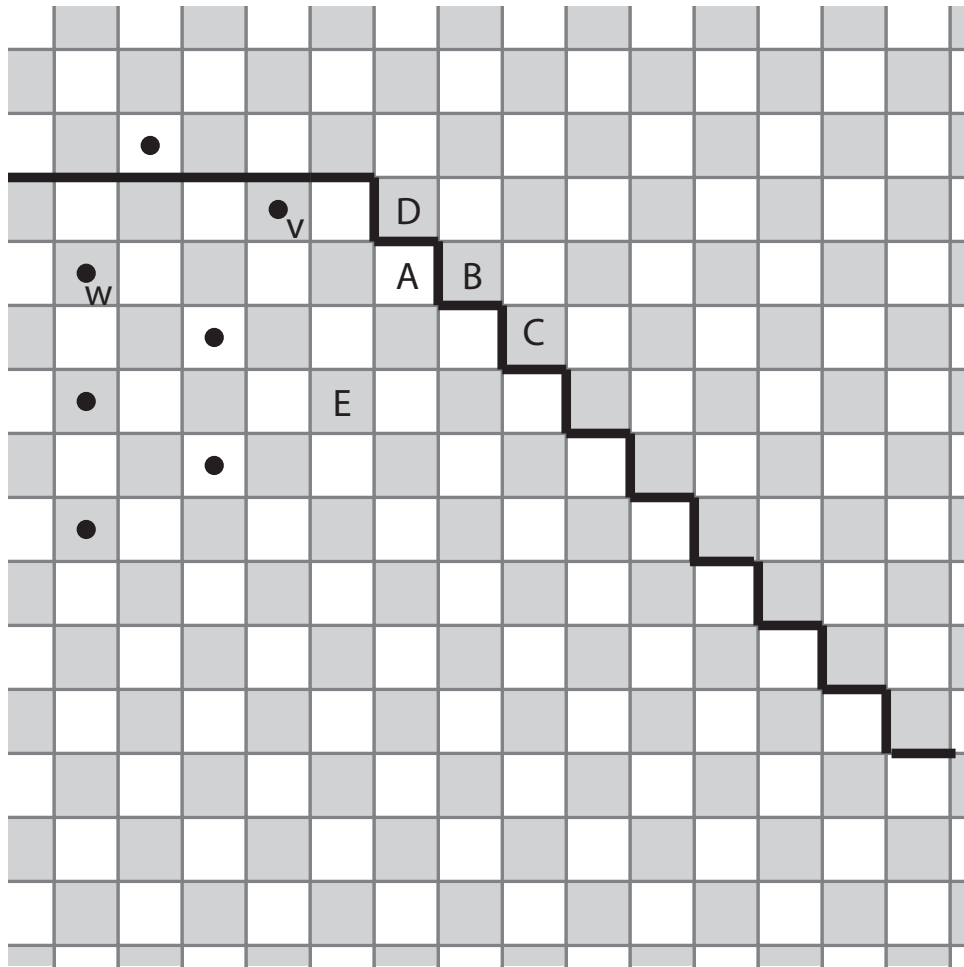


Figure 9.12: Case 4 (b) (i).



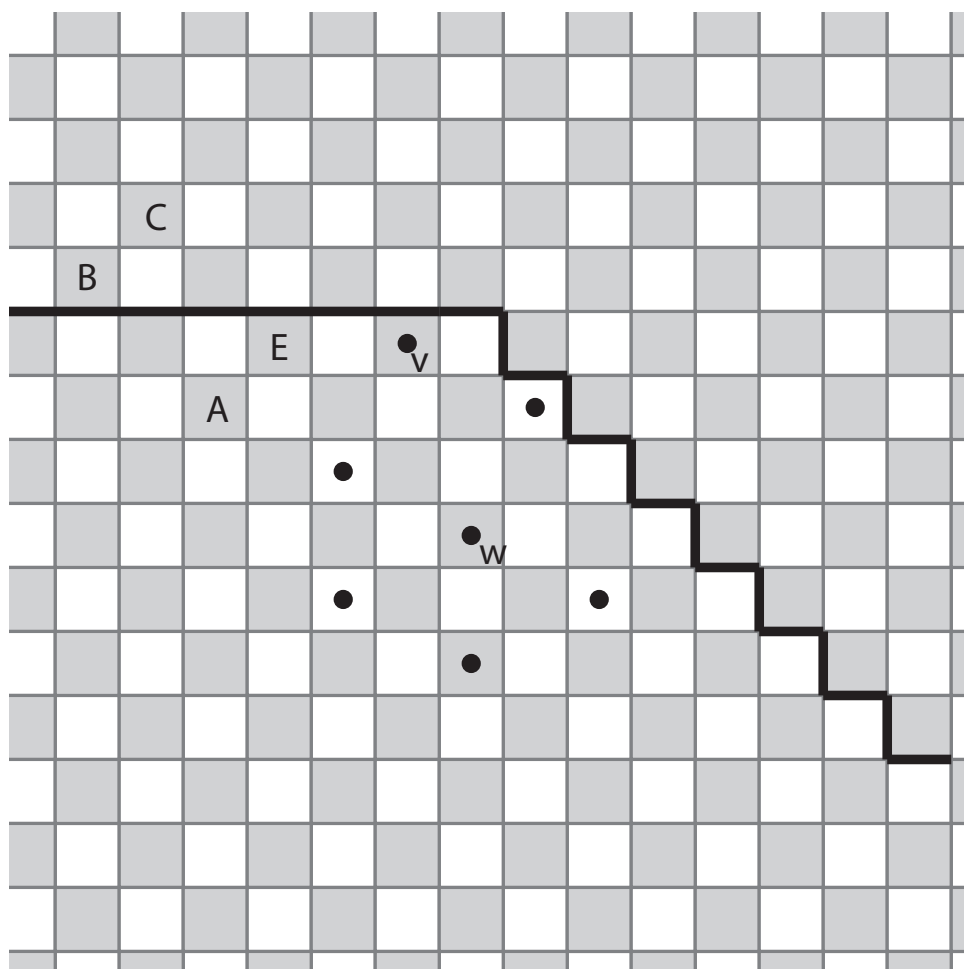


Figure 9.13: Case 4 (b) (ii).

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